

SEMI-ANNUAL STATUS REPORT

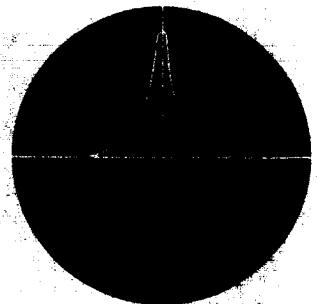
VOLUME II

RESEARCH IN AND APPLICATION OF
MODERN AUTOMATIC CONTROL THEORY
TO NUCLEAR ROCKET DYNAMICS AND CONTROL

Prepared Under Grant NsG-490
National Aeronautics and Space Administration

UNPUBLISHED PRELIMINARY DATA
ENGINEERING RESEARCH LABORATORIES

COLLEGE OF ENGINEERING
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA



FACILITY FORM 602

N 65-21462
(ACCESSION NUMBER)

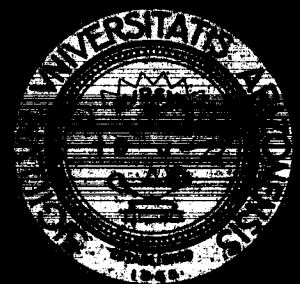
165
(PAGES)

CR-62187
(NASA CR OR TMX OR AD NUMBER)

(THRU)

1
(CODE)

10
(CATEGORY)



GPO PRICE \$ _____

OTS PRICE(S) \$ _____

Hard copy (HC) \$ 5.00

Microfiche (MF) \$ 1.00

SEMI-ANNUAL STATUS REPORT

Volume II

RESEARCH IN AND APPLICATION OF
MODERN AUTOMATIC CONTROL THEORY
TO NUCLEAR ROCKET DYNAMICS AND CONTROL

Prepared Under Grant NsG-490
National Aeronautics and Space Administration
(Lynn E. Weaver, Project Director)

by

James L. Melsa
Donald G. Schultz

Nuclear Engineering Department
The University of Arizona
Tucson, Arizona

March 1965

Engineering Research Laboratories
The University of Arizona
College of Engineering
Tucson, Arizona

PREFACE

This report represents the completion of one phase of the study of sub-optimal control methods, a study sponsored by the National Aeronautics and Space Administration under Grant NsG-490 on research in and application of modern automatic control theory to nuclear rocket dynamics and control. The report is intended to be a self-contained unit and therefore repeats some of the work presented in previous status reports.

Portions of the work were submitted to the Department of Electrical Engineering at the University of Arizona in partial fulfillment of the requirements for the degree of Doctor of Philosophy; this dissertation research was supported in part by the National Science Foundation under Grant GP-2237.

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	v
ABSTRACT	vii
CHAPTER 1: INTRODUCTION AND ORGANIZATION	1
1.1 Introduction	1
1.2 Organization of the Work	2
CHAPTER 2: THE MINIMUM PRINCIPLE	5
2.1 Introduction	5
2.2 Notation	5
2.3 Formulation of the Basic Optimization Problem	6
2.4 Minimum Principle	10
CHAPTER 3: THE SECOND METHOD OF LIAPUNOV	16
3.1 Introduction	16
3.2 Definitions	17
3.3 Stability Theorem	22
CHAPTER 4: CLOSED-LOOP OPTIMAL CONTROL VIA THE SECOND METHOD	26
4.1 Introduction	26
4.2 Background	27
4.3 Optimality Theorem	29
4.4 Hamilton-Jacobi Equation	33
CHAPTER 5: EIGENVECTOR SCALAR PRODUCT SOLUTIONS	36
5.1 Introduction	36
5.2 Modification of Hamilton-Jacobi Equation	36
5.3 Eigenvector Scalar Products	42
5.4 Bounds on Transition Time	50

TABLE OF CONTENTS--Continued

	Page
CHAPTER 6: SUB-OPTIMAL CONTROL WITH NON-SINGULAR CONTROL MATRIX	59
6.1 Introduction	59
6.2 Sub-Optimal Control of Second-Order Systems . .	60
6.3 Generalization	73
6.4 Discussion of the Method	79
CHAPTER 7: SUB-OPTIMAL CONTROL WITH SINGULAR CONTROL MATRIX	83
7.1 Introduction	83
7.2 The Bang-Coast Method	87
7.3 The Unbounded Control Method	92
7.4 The Switched Control Method	95
7.5 Combination of Methods	114
CHAPTER 8: APPLICATIONS TO NUCLEAR ROCKET CONTROL PROBLEMS	120
8.1 Introduction	120
8.2 Time-Optimal Control of Single-Precursor Group Neutron Kinetics	123
8.3 Time-Optimal Control of a Simple Nuclear Rocket Engine	127
8.4 Summary	133
CHAPTER 9: SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH	135
9.1 Summary	135
9.2 Suggestions for Future Research	140
APPENDIX A: SUB-OPTIMAL CONTROL METHODS USING THE SECOND METHOD	142
A.1 Introduction	142
A.2 Estimation of Transient Behavior	143
A.3 Kalman-Bertram Method	147
A.4 The Nahi Method	150
LIST OF REFERENCES	154

LIST OF ILLUSTRATIONS

Figure	Page
3.1 Graphical Representation of Stability Definitions .	21
3.2 Surfaces of $V(x) = \text{Constant}$	25
5.1 Typical Isochrones	52
5.2 Surfaces of Constant $V(x)$ for Eigenvector Scalar Product Solutions	55
5.3 A Typical S_2 Surface Bounding the Optimal S_0 Surface	57
6.1 Typical Eigenvector Scalar Product Optimal Trajectory	62
6.2 Typical Optimal and Non-Optimal Controls	64
6.3 Representation of a Vector in Terms of the Optimal Control Vectors	67
6.4 Closed-Loop, Sub-Optimal Control System	71
6.5 Third-Order Example for Sub-Optimal Control with Non-Singular Control Matrix	77
7.1 R Subset for a Second-Order System	98
7.2 Second-Order, Time-Optimal Trajectory	100
7.3 Sub-Optimal Policy for Second-Order Systems with Scalar Control	102
7.4 Sub-Optimal Policy for n-th Order Systems with $r = n-1$	110
7.5 Time Response of a Third-Order System with Two Control Variables	113
7.6 Time Response of a Third-Order System with Scalar Control	117

LIST OF ILLUSTRATIONS--Continued

Figure	Page
8.1 Approximation of a Typical Control Region	124
8.2 Approximation of the Control Region by a Norm Constraint	132
A.1 Estimation of Settling Time	145

ABSTRACT

In this work the Second Method of Liapunov is used as a basis for developing a method of closed-loop, approximately time-optimal control of linear systems with bounded control norm.

As a first step in this development, the Second Method is combined with the minimum principle to create a new approach to the optimization problem. The approach centers on the solution of a first-order partial differential equation, the Hamilton-Jacobi equation. Although it is not possible to solve this equation in general, a special class of solutions is shown to exist. These solutions, called eigenvector scalar products, form the basis of an effective closed-loop, sub-optimal control method.

The eigenvector scalar product solutions are used first for the control of systems in which the control matrix is non-singular. The method is based on the concept of finding a control vector of unit magnitude such that the value of each of the eigenvector scalar product solutions is equal. This control vector transfers the system to the origin in a finite time. In the absence of disturbances, the control vector, once obtained, remains constant until the system reaches the origin. In designing a closed-loop control system using this method, it is necessary that the

controller-computer solve only algebraic equations, thus allowing continuous control.

By the use of the eigenvector scalar products, it is possible to find in an unusually simple manner surfaces which bound the optimal isochrones. These bounds on the optimal isochrones enable one to judge the quality of a sub-optimal control system. The inability to find such bounds on the optimal isochrones has been a serious difficulty in designing sub-optimal systems in the past.

When the control matrix is allowed to be singular, it is no longer possible to develop a single method to handle all problems. Hence, two methods are developed, each of which has restrictions. The first method, called the Bang-Coast Method, is based on the concept of equating only r of the eigenvector scalar product solutions rather than all n . The second method, called the Switched Control Method, is based on the concept of driving the system to the r -dimensional subset of the state space in which all n eigenvector scalar product solutions can be equated. These two methods can be combined to further increase their range of applicability. In this manner, effective sub-optimal control of systems with singular control matrices can be obtained.

A procedure is outlined for applying these sub-optimal methods to some practical control problems and is

illustrated by application to two simple nuclear rocket control problems.

In summary, there are three major contributions of this work. First, a special class of solutions of the Hamilton-Jacobi equation, called eigenvector scalar products, is shown to exist. Second, by the use of the eigenvector scalar product solutions, a method of judging the quality of a sub-optimal system by bounding the optimal isochrones is developed. Third, the eigenvector scalar product solutions are used to develop several methods of closed-loop, sub-optimal control. The procedure to be followed in each method is systematically presented, and in each case, the method is shown to represent an effective compromise between system complexity and speed of response.

CHAPTER 1

INTRODUCTION AND ORGANIZATION

1.1 Introduction

The problem of controlling a system such that its performance approximates in some sense a desired performance has been of importance for a long time. A natural outgrowth of this interest is the optimal control problem: controlling a system in such a manner that its performance is the best possible.

Within the last few years, several rather elegant, general methods of solving the optimal control problem have been presented. Notable among these is the maximum principle of Pontryagin. In general, these methods involve unwieldy computations for all but trivial problems. Also in many cases, the control once obtained is of open-loop nature; that is, it is valid for only one initial condition and no disturbances.

The difficulties associated with these methods have led to a growing gap between theoretical and practical control work. To fill this gap, there has been an ever-increasing development of special techniques for special problems which generally lead to sub-optimal control, control which is acceptably close to the true optimal but practicable.

In this work the Second Method of Liapunov is used as a basis for developing such a method for closed-loop optimal control of linear systems with a bounded control norm. This method centers on the solution of a partial differential equation which is equivalent to the Hamilton-Jacobi equation. A special class of solutions, called eigenvector scalar products, is shown to exist. These solutions are combined to form a sub-optimal control method which provides a practical compromise between system complexity and speed of response for a large, although limited, class of systems.

1.2 Organization of the Work

This work consists of three basic parts. The first part comprising Chapters 1, 2, and 3 is introductory in nature. Following the introductory material in this chapter, the basic optimization problem to be considered is formulated in Chapter 2. Chapter 2 also contains a brief review of a modified form of the maximum principle which has been termed the minimum principle. In Chapter 3 a brief introduction to the Second Method of Liapunov is presented in order to make the work a self-contained unit.

Chapters 4 and 5 form the second part, the theoretical heart of this work. In Chapter 4, the Second Method is combined with the minimum principle to develop another approach to the basic optimization problem. It is

demonstrated that solving the basic optimization problem is equivalent to solving a first-order partial differential equation which is identical to the Hamilton-Jacobi equation. Although no general method of solving this equation is known, a special class of solutions is shown to exist. This class of solutions, called eigenvector scalar products, is developed and discussed in detail in Chapter 5.

The third part, consisting of Chapters 6, 7, and 8, is the practical portion of the work. In Chapter 6, the eigenvector scalar product solutions are combined to form an effective sub-optimal control method for systems in which the control matrix is non-singular. In this form, the sub-optimal control method provides an effective solution to a limited class of practical systems.

In Chapter 7, this method is extended to the case where the control matrix is singular. Although it is not possible to solve all problems in this class, a solution is obtained for a large number of practical systems. In Chapter 8, a procedure is outlined for applying the sub-optimal control methods developed in Chapters 6 and 7 to some practical control problems. In particular, attention is directed toward nuclear rocket control.

Chapter 9 contains a discussion of the concepts introduced and several ideas for further research. Examples are presented throughout the work whenever they can serve to better illustrate a point.

A basic knowledge of vector and matrix algebra is expected of the reader, as well as an understanding of the state variable method of formulating control problems. Although a brief review of the minimum principle and the Second Method are presented, the reader who is not familiar with these methods may wish to consult some of the suggested references for a more introductory presentation.

CHAPTER 2

THE MINIMUM PRINCIPLE

2.1 Introduction

This chapter consists of two basic parts. First, the basic optimization problem of this work is formulated, including all necessary definitions and notation. Second, a brief description of the minimum principle method for solving this problem is presented. Since extensive accounts of this method may be found in the literature (Kalman 1961, Pontryagin et al. 1962), only the aspects pertinent to the particular problem of this work are included. Those familiar with the minimum principle may wish to skip Section 2.4.

2.2 Notation

In this section, the notation which is used throughout is explained. In general, the state space approach is employed, utilizing vector-matrix formulation. Vectors are indicated by lower case Roman letters such as x , u . Lower case Roman letters are also used to designate scalars when there is no chance of confusion, such as in subscripts. The components of a vector are indicated by subscripted lower case Roman letters; therefore $x = (x_1, x_2, \dots, x_n)$. Particular vectors are indicated by superscripts; therefore $x^1 = (x_1^1, x_2^1, \dots, x_n^1)$.

Matrices are designated by underlined upper case Roman letters such as A, B; scalars are designated by upper case Roman letters or Greek letters. The transpose of a matrix or vector is designated by a prime; therefore x' is the transpose of the vector x .

The notation $\partial L(x)/\partial x$ is used to indicate a vector whose components consist of the partial derivatives of $L(x)$; thus $\partial L(x)/\partial x = (\partial L(x)/\partial x_1, \dots, \partial L(x)/\partial x_n)$. The notation $\nabla L(x)$ is also used when the differentiation is with respect to x ; thus, $\nabla L(x) = \partial L(x)/\partial x$.

2.3 Formulation of the Basic Optimization Problem

The state of the control system is described at any instant of time by n real numbers, x_1, x_2, \dots, x_n . The behavior (or motion) of the system as a function of time may then be described by n real functions of time, $x_1(t), x_2(t), \dots, x_n(t)$. These variables, called state variables, are the components of the state vector $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$.

The motion of the system is controlled by a set of r real valued control variables, $u_1(t), u_2(t), \dots, u_r(t)$, which are the components of the control vector, $u(t)$. The set of all possible values of u is called the control region, U , a subset of an r -dimensional Euclidean space. In most practical applications, U is closed and bounded.

For the present work U consists of the set of all u such that $\|Du\|^2 \leq \alpha^2$ where D is a non-singular matrix and α is

a real constant. However, by a simple change of variables $w = \alpha^{-1}Du$, $\|Du\|^2 \leq \alpha^2$ becomes $\|w\|^2 \leq 1$. Hence there is no loss of generality in considering D to be the identity matrix and α to be unity. Thus U is the set of all u such that $\|u\|^2 \leq 1$. If $u(t) \in U$ and is, in addition, piecewise continuous, then $u(t)$ is called an admissible control.

In practical terms this constraint on the control norm implies that the total control effort is limited, as for example if all of the control effort were obtained from a limited power source. This constraint should be contrasted with a constraint on the magnitude of each control variable, i.e. $|u_i| < 1$, $i = 1, 2, \dots, n$. In this latter case the amount of control which may be exerted on the system at any input point is limited. Although both cases are of practical importance, only the case of limited control effort is considered in this work.

The only systems to be considered here are ones for which the laws of motion may be written as a set of n first-order linear equations.

$$\dot{x}_i = \sum_{j=1}^n a_{ij}x_j + \sum_{k=1}^r b_{ik}u_k \quad i = 1, 2, \dots, n \quad (2.1)$$

They may also be written in vector-matrix notation.

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}u \quad (2.2)$$

It is assumed that corresponding to every admissible control $u(t)$ and every initial condition $x^0 = x(t_0)$, that

the motion of the system is defined uniquely by the solution of Equation (2.2). This solution is called the solution (or motion) of the system corresponding to the control $u(t)$ for the initial condition x^0 .

An admissible control is said to transfer the system from x^0 to x^1 if the solution corresponding to that control and the initial condition x^0 is defined for $t_0 \leq t \leq t_1$ and reaches x^1 at the time t_1 .

Since, in general, there may be many admissible controls which transfer the system from x^0 to x^1 , the question which naturally arises is, "Which admissible control, in addition to transferring the system from x^0 to x^1 , minimizes some cost functional

$$J = \int_{t_0}^{t_1} L(x(t))dt \quad (2.3)$$

where $L(x)$ is a real and positive-valued scalar function of the state vector?"

It should be noted that for fixed points, the transition time, $t_1 - t_0$, is not fixed but is dependent on the particular control used. One example of particular importance is the case when $L(x) = 1$ and the cost functional, J , reduces to $t_1 - t_0$, the transition time. This is the familiar time-optimal problem with a constraint on the control norm which is treated in detail in later chapters.

A control which transfers the system from x^0 to x^1 while minimizing the cost functional is called an optimal control corresponding to a transition from x^0 to x^1 . For convenience, x^1 is considered to be the origin for the rest of this work.

The optimal control may be found in two different forms. First, the control variables may be obtained as functions of time during the transition interval $t_1 - t_0$ for a given initial condition x^0 . This is called open-loop control, since no information concerning the system state is needed or used during the transition interval.

Second, the control variables may be determined as explicit functions of the system state, i.e., $u = u(x)$. This is called closed-loop control, since knowledge of the system state is used during the transition interval. The advantages of closed-loop control are well established in the literature (Horowitz 1963, Truxal 1955), and therefore only three points are mentioned here. First, feedback or closed-loop operation reduces the effect of system parameter variations. Second, feedback operation minimizes the effect of external disturbances. Third, in many practical cases, the equations of motion are known only approximately. By the use of closed-loop control, variations in the system's motion due to these inaccuracies can be minimized. Thus it appears obvious that not only should one seek optimal

control, but, in general, one should seek closed-loop optimal control.

The fundamental problem may then be stated in the following form. Given a linear system whose laws of motion are described by Equation (2.2), it is desired to find an optimal, closed-loop, admissible control corresponding to a transition from x^0 to the origin with a cost functional of the form of Equation (2.3). Additional assumptions concerning the system and the cost functional are made in later chapters.

The next section presents the basic formulation and theorems of the minimum principle, a method for obtaining an open-loop solution of the above problem.

2.4 Minimum Principle

The concept of the minimum principle was first introduced by Kalman (1961) as a minor modification of the maximum principle developed by Pontryagin and his students (Boltyanskii, Gamkrelidze, and Pontryagin 1960). The essential differences between the two approaches are noted later. The minimum principle is a logical extension of the classical calculus of variations and provides a broad and unifying approach to a wide variety of variational and optimal control problems. Only those aspects of the theory which are pertinent to the problem of the preceding section are presented here.

As the first step in the minimum principle approach, a new set of n variables, p_i , are adjoined to the state variables, x_i , of the system. These new variables, called adjoint variables, are defined by the following set of differential equations, the adjoint equation.

$$\dot{p}_i = -\frac{\partial}{\partial x_i} [p' \dot{x} + L(x)] \quad i = 1, 2, \dots, n \quad (2.4)$$

Next a scalar function H analogous to the Hamiltonian is defined by

$$H(x, p, u) = p' \dot{x} + L(x) \quad (2.5)$$

For convenience, H is referred to simply as the Hamiltonian. It can be readily verified that Equations (2.2) and (2.4) can be rewritten in terms of $H(x, p, u)$ in the following system of equations which are analogous to the Hamiltonian canonic equations.

$$\begin{aligned} \dot{x}_i &= \frac{\partial}{\partial p_i} (H(x, p, u)) \\ \dot{p}_i &= -\frac{\partial}{\partial x_i} (H(x, p, u)) \quad i = 1, 2, \dots, n \end{aligned} \quad (2.6)$$

For fixed values of x and p , H becomes a function of the control vector u . The greatest lower bound of this function with respect to admissible controls $u \in U$ will be denoted by H^0 ; therefore

$$H^0(x, p) = \inf_{u \in U} H(x, p, u) \quad (2.7)$$

If the continuous function H actually assumes its lower bound on U , then H^0 will be the minimum of H on U . This is true for all problems in this work; hence

$$H^0(x,p) = \min_{u \in U} H(x,p,u) \quad (2.8)$$

The following theorem presents a necessary condition for the optimality of a control u .

Theorem 2.1 Let $u^0(t)$, $t_0 \leq t \leq t_1$, be an admissible control such that the corresponding motion $x(t)$ which begins at the point x^0 at time t_0 reaches the origin at time t_1 . In order that $u^0(t)$ and $x(t)$ be optimal, it is necessary that there exist a nonzero continuous vector function $p(t)$ corresponding to $u^0(t)$ and $x(t)$ such that:

- 1) for every t , $t_0 \leq t \leq t_1$, the function $H(x,p,u)$ of the variable $u \in U$ attains its minimum at the point $u = u^0(t)$:

$$H(x,p,u) = H^0(x,p)$$

- 2) for every t , $t_0 \leq t \leq t_1$, the function $H^0(x,p)$ is identically zero:

$$H^0(x(t),p(t)) = 0$$

This theorem formulated in terms of the minimum principle is equivalent to a theorem of the maximum principle initially proven by Pontryagin (1962). In the maximum principle formulation, the sign preceding $L(x)$ in both Equations (2.4) and (2.5) is negative. Because of this

change, it is necessary to consider the least upper bound of $H(x,p,u)$, rather than the greatest lower bound. Hence H is maximized rather than minimized. Although the use of the maximum principle is more common in the literature, the use of the minimum principle is more convenient for the development of Chapter 4, and thus it is employed here.

For the problem presented in the preceding section, the Hamiltonian is given by

$$\begin{aligned} H(x,p,u) &= p'(\underline{A}x + \underline{B}u) + L(x) \\ &= p'\underline{A}x + p'\underline{B}u + L(x) \end{aligned} \quad (2.9)$$

The adjoint equation may then be developed by use of Equation (2.6).

$$\dot{p} = -\underline{A}'p - \nabla L(x) \quad (2.10)$$

The next step is the minimization of $H(p,x,u)$ with respect to $u \in U$. Since the middle term on the right side of Equation (2.9) is the scalar product of two vectors, $p'\underline{B}$ and u , $H(x,p,u)$ is minimized by making the direction of u opposite to $\underline{B}'p$ and making the magnitude of u as large as possible. However, the norm of u is required to be less than or equal unity in order for u to be an admissible control. Hence, u is selected to be a vector with unit norm (length) and direction opposite $\underline{B}'p$:

$$u^0 = - \frac{\underline{B}'p}{\|\underline{B}'p\|} \quad (2.11)$$

Substituting u as given by Equation (2.11) into the Equations (2.2) and (2.9), the following set of coupled first-order ordinary differential equations are obtained.

$$\dot{x} = \underline{A}x + \frac{\underline{B}\underline{B}'p}{\|\underline{B}'p\|} \quad (2.12)$$

$$\dot{p} = -\underline{A}'p - \nabla L(x) \quad (2.13)$$

with the boundary conditions $x(t_0) = x^0$ and $x(t_1) = 0$ and the auxiliary condition $H^0(x, p) = 0$.

The difficulties inherent in the minimum principle approach are now obvious. First, the simultaneous solution of Equations (2.12) and (2.13) is not elementary, since both equations are in general nonlinear. Since the adjoint equation has no boundary conditions, while the system equations have $2n$ boundary conditions, the so-called "two-point" boundary value problem is created and numerical solution is normally necessary. Second, the control as determined by the minimum principle is open-loop control, i.e., $u = u(t)$ not $u(x)$.

Another method for attacking the basic optimization problem of the preceding section is presented in Chapter 4. The method is based on both the Second Method of Liapunov and the minimum principle and attempts to remove or alleviate the difficulties mentioned above. In particular, the control vector is found as a function of the state variables, i.e., closed-loop control. However, before proceeding to

that development, it is necessary to present some of the basic definitions and theorems of the Second Method.

CHAPTER 3

THE SECOND METHOD OF LIAPUNOV

3.1 Introduction

The Second Method of Liapunov provides the most general approach to the stability of dynamic systems whose laws of motion are described by ordinary linear or nonlinear differential equations. This chapter presents a brief review of the basic concepts and definitions of the Second Method. Only those portions of the theory which are directly applicable to the problem at hand are discussed. The reader is directed to the literature for a more complete presentation (Hahn 1963, LaSalle 1960, LaSalle and Lefschetz 1961, Schultz 1962).

In this chapter, the dynamic systems under consideration are assumed to be autonomous and describable in state variable form as n first-order differential equations of the form

$$\dot{x}_i = f_i(x) \quad i = 1, 2, \dots, n \quad (3.1)$$

In matrix notation, this may be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (3.2)$$

Such a system is called autonomous. It is obvious that for

closed-loop control the system of Equation (2.2) is of this form since it becomes

$$\begin{aligned}\dot{x} &= \underline{A}x + \underline{B}u(x) \\ &= f(x)\end{aligned}$$

The equilibrium state being investigated is assumed to be located at the origin. This is actually no restriction, since any equilibrium point may always be translated by simple linear change of variables to the origin. Again the system discussed in Chapter 2 satisfies this assumption, since the control is always chosen such as to drive the system to the origin.

This chapter consists of three parts. First, the definitions of definiteness and stability are presented. Second, a modified Liapunov stability theorem is stated without proof. Third, this stability theorem is given a geometric interpretation.

3.2 Definitions

The concepts of definiteness play an important role in the stability theorems. The following definitions, which follow Malkin (1958), are of interest here.

Definition 3.1 Positive (Negative) Definite

A scalar function, $V(x)$, is positive (negative) definite if for $\|x\| \leq \alpha$ $V(x) > 0$ (< 0) for all $x \neq 0$ and $V(0) = 0$.

Definition 3.2 Positive (Negative) Semidefinite

A scalar function, $V(x)$ is positive (negative) semidefinite if for $\|x\| \leq \alpha$ $V(x) \geq 0$ (≤ 0) for all $x \neq 0$ and $V(0) = 0$.

Definition 3.3 Indefinite

A scalar function, $V(x)$, is indefinite if no matter how small α is chosen, $V(x)$ may assume both positive and negative values for $\|x\| \leq \alpha$.

If in the above definitions α may be made arbitrarily large, the definitions hold in the whole space. This is the case with all of the scalar functions to be discussed in the following chapters.

A few examples serve to clarify these definitions.

The function

$$V(x) = (x_1)^2 + (x_2)^4$$

is positive definite if the system is second-order, but is only semidefinite if the system is of higher order, since for $x_1 = x_2 = 0$, $V(x)$ is zero independent of x_3, x_4, \dots . On the other hand the function

$$V(x) = (x_1 + x_2)^2$$

is semidefinite even for second-order systems, since if $x_1 = -x_2$, $V(x)$ is zero even though x is not equal to zero.

The function

$$V(x) = x_1 + x_2$$

is obviously indefinite independent of the order of the system.

One class of scalar functions of particular importance is a quadratic form. In this case $V(x)$ may be written in the form

$$V(x) = x' \underline{C} x$$

where \underline{C} is a constant square matrix. Usually if $V(x)$ is a quadratic form, the definiteness of $V(x)$ is attributed to \underline{C} . Hence one speaks of a positive definite matrix.

Closely related to the concept of definiteness is the concept of a simple closed surface (or curve). A surface is said to be simple if it does not intersect itself and closed if it intersects all paths that lead from the origin to infinity. The reader is reminded that it is assumed that the equilibrium state is at the origin. Hence a simple closed surface is topologically equivalent to the surface of an n -dimensional sphere. Letov (1961) has shown that if a scalar function, $V(x)$, is positive definite and, in addition, is radially unbounded, i.e., $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then the set of all points x such that $V(x) = K$, a positive constant, is a simple closed surface. In addition, the surface $V(x) = K_1$ lies entirely inside the surface $V(x) = K_2$ whenever $K_1 < K_2$.

There are many types of stability that have been defined for systems that may be described by Equation (3.2).

In the case of linear systems, almost all of these definitions are equivalent. For nonlinear systems, this is not true. However, for this work, only stability in the sense of Liapunov and asymptotic stability are of interest. Hence only these types of stability are defined. Let $S(\alpha)$ be the spherical region of radius $\alpha > 0$ around the origin, i.e., $S(\alpha)$ consists of all points x such that $\|x\| < \alpha$.

Definition 3.4 Stable in the Sense of Liapunov

The origin is stable in the sense of Liapunov, or simply stable, if corresponding to every number $\epsilon > 0$ there exists a number $\delta(\epsilon) > 0$ such that solutions starting in $S(\delta)$ will remain in $S(\epsilon)$ ever after.

Definition 3.5 Asymptotically Stable

If the origin is stable and, in addition, every solution starting in $S(\delta)$ not only stays in $S(\epsilon)$ but tends toward the origin as time increases indefinitely, then the origin is asymptotically stable.

Definition 3.6 Unstable

The origin is unstable if for some $\epsilon > 0$ and any $\delta > 0$, no matter how small, there is always a point x in $S(\delta)$ such that a solution starting from that point leaves $S(\epsilon)$.

A graphical representation of these definitions is shown in Figure 3.1 for a two-dimensional case.

The definitions emphasize the local character of stability for nonlinear systems, since the region $S(\delta)$ may

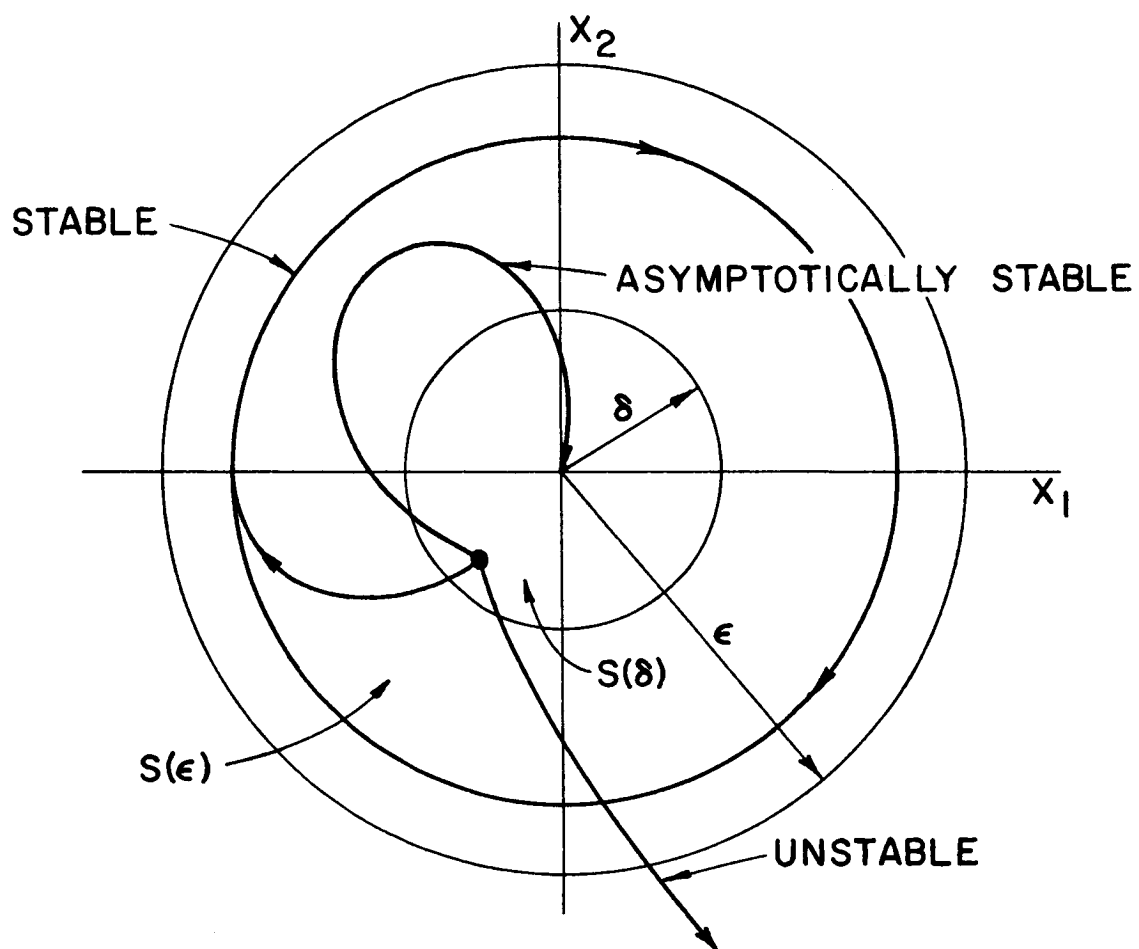


Figure 3.1 Graphical Representation of Stability Definitions

be arbitrarily small. If the region $S(\delta)$ includes the entire space, the definitions are called global. In the chapters which follow, the main interest is in global asymptotic stability, since the systems are linear.

3.3 Stability Theorem

As was the case with definitions of stability, there are many stability theorems which constitute the Second Method. Since the major concern of this work is not stability, only one theorem is presented here. This stability theorem, due to LaSalle (1960), differs from the original Liapunov theorem in that $\dot{V}(x)$ is allowed to be semidefinite, as long as it is not zero on a solution of the system, other than the origin. In the original theorem, $\dot{V}(x)$ was required to be negative definite.

Before stating this theorem, it is convenient to define a special class of scalar functions, Liapunov functions.

Definition 3.2 Liapunov Function

A positive definite scalar function, $V(x)$, with continuous first partial derivatives, is called a Liapunov function if its total time derivative, $\dot{V}(x)$ is negative semidefinite.

Since $V(x)$ has continuous first partials, the chain rule may be used to obtain $\dot{V}(x)$.

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V(x)}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V(x)}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial V(x)}{\partial x_n} \frac{dx_n}{dt} \\ &= \sum_{i=1}^n \frac{\partial V(x)}{\partial x_i} \dot{x}_i\end{aligned}$$

This may be written with the use of the notation $\nabla V(x)$ as

$$\dot{V}(x) = \nabla V'(x) \dot{x} \quad (3.3)$$

In later chapters, functions which are only positive semidefinite with negative semidefinite time derivatives are also called Liapunov functions for simplicity even though they do not satisfy the strict definition. In terms of the Liapunov function, the stability theorem may be stated in the following form.

Theorem 3.1 Stability Theorem

If there is a Liapunov function, $V(x)$, such that $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ (radially unbounded) and if $\dot{V}(x)$ is not identically zero along any solution of Equation (3.2) other than the origin, then the system is globally asymptotically stable.

The basic concept of the Second Method is now evident: by proper selection or generation of a Liapunov function, it is possible to determine the stability of a nonlinear dynamic system without any knowledge of the solutions of the system equation. It is perhaps of value to investigate the stability theorem from a geometric viewpoint.

Since $V(x)$ is positive definite, and radially unbounded $V(x) = K$, a constant, becomes a family of concentric closed surfaces surrounding the origin such that the surface $V(x) = K_1$ lies inside $V(x) = K_2$ whenever $K_1 < K_2$. Figure 3.2 shows a graphical picture for the two-dimensional or second-order case. Since both $V(x)$ and $\dot{V}(x)$ are implicit functions of time and $\dot{V}(x)$ is required to be non-positive, the state of the system must be found on successively "smaller" $V(x) = K$, a constant, surfaces or must remain stationary. But $\dot{V}(x)$ cannot be zero on any solution except $x = 0$; therefore the state of the system cannot remain stationary. Hence, the system trajectory must move toward the origin.

Three features of the Second Method should be noted. First, the method provides only sufficient conditions for stability; hence if a system does not satisfy the stability theorem, no conclusion may be drawn relative to system stability. Second, the converse of the stability theorem has been proven. Therefore if the system is stable, a Liapunov function must exist. Third, the Liapunov function is not unique, which is one of the most powerful features of the Second Method. No longer is one searching for a single unique solution to the differential equation but rather for one out of many Liapunov functions. However, because the method provides only sufficient conditions, some Liapunov functions may provide a better answer than others.

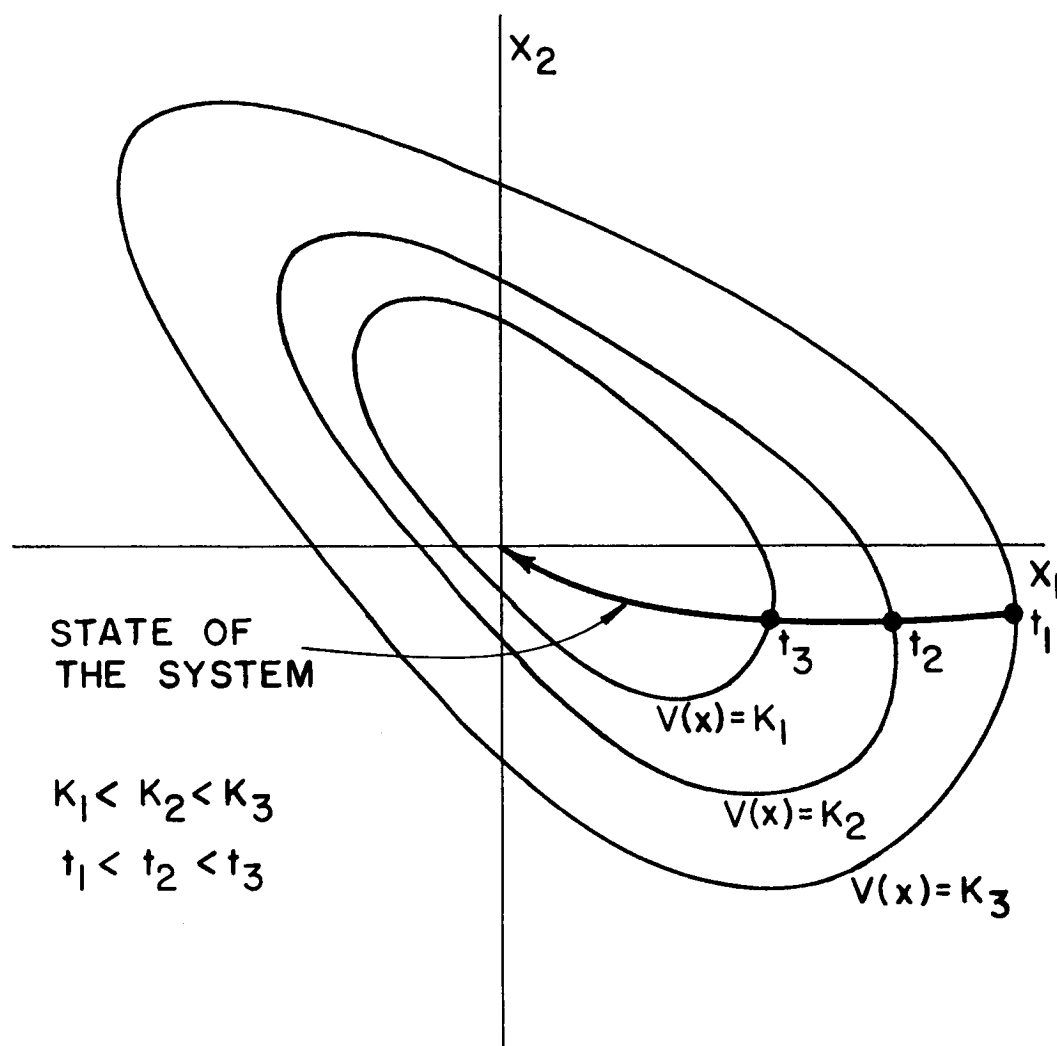


Figure 3.2: Surfaces of $V(x) = \text{Constant}$

CHAPTER 4

CLOSED-LOOP OPTIMAL CONTROL VIA THE SECOND METHOD

4.1 Introduction

In Chapter 2 the basic optimization problem was presented. This was followed by one method of obtaining an open-loop solution of the problem, the minimum principle. In this chapter another method of attacking the basic optimization problem is presented. This method, based on the Second Method of Liapunov and the minimum principle, yields closed-loop control.

In the next section a brief discussion of the background for the use of the Second Method is presented. This is followed by an optimality theorem and its proof. It is demonstrated that solving the basic optimization problem is equivalent to solving a first-order partial differential equation which is identical to the Hamilton-Jacobi equation. Since no general method of solving this equation is known, the approach presented here has not solved the problem but has rather formulated the problem into a new framework. In this framework, a special class of solutions, called eigenvector scalar products, is shown to exist in the next chapter. From these solutions, a method is developed for designing effective closed-loop, sub-optimal control for a large, although limited, class of systems.

It should be noted that the results of this chapter are not new, although the method of deriving them is. As is shown in the last section of this chapter, the results could have been derived directly from the Hamilton-Jacobi equation. In effect, a special case of the Hamilton-Jacobi equation is derived in this chapter. It is felt that carrying out the development in this manner adds greater insight into the relation between the Second Method and optimal control.

4.2 Background

The use of the Second Method of Liapunov for the design of optimal systems has been suggested by several authors (Johnson 1963, Kalman and Bertram 1960, LaSalle 1962, Letov 1961, and Nahi 1964). Unfortunately, almost all of these methods have three basic problems: 1) they are approximate, 2) either no estimate of the approximation error is possible, or the estimate is overly conservative, and 3) it is necessary to choose a $V(x)$ for which no general procedure is presented. Hence these methods were never widely accepted. (A brief resume of several of these methods can be found in the Appendix.)

Nahi (1964) has recently presented a procedure for using the Second Method to obtain time-optimal control. However, Nahi was only able to find solutions for a rather

restricted class of systems. It is shown later that Nahi's method is a special case of the method presented here.

The determination of \dot{V} from $V(x)$ was discussed in Chapter 3; the result is repeated here for reference.

$$\dot{V} = \nabla V'(x)\dot{x} \quad (4.1)$$

Now substituting Equation (2.2) for \dot{x} , one obtains

$$\dot{V} = \nabla V'(x)\underline{A}x + \nabla V'(x)\underline{B}u \quad (4.2)$$

Thus \dot{V} becomes a function of both the control and state vectors for a given $V(x)$. In the following discussion the notation $\dot{V}(x,u)$ is used to indicate this dependence on both u and x .

In 1960 Kalman and Bertram presented a method for designing approximately time-optimal control systems. Their method was based on the knowledge that for a closed, bounded control region, the control vector is always on the boundary. They suggested minimizing $\dot{V}(x,u)$ with respect to all admissible controls based on the argument that this would make $V(x)$ approach zero most rapidly and hence the system would reach the origin in minimum time. This method suffers from all of the disadvantages noted above and therefore has not been widely employed. However, the concept of minimizing $\dot{V}(x,u)$ is valuable and is used below.

Retaining the idea of minimizing $\dot{V}(x,u)$ for the moment, consider the implication of setting $\dot{V}(x) = -L(x)$.

Since $L(x)$ was required to be at least positive semidefinite, $\dot{V}(x)$ is thus of the proper nature. Then $V(x)$ becomes equivalent to the cost functional:

$$V(x(t_1)) - V(x(t_0)) = \int_{t_0}^{t_1} \dot{V}(x) dt = \int_{t_0}^{t_1} -L(x) dt \quad (4.3)$$

Hence surfaces of constant $V(x)$ become surfaces of constant cost.

The combination of these two concepts suggests the idea of setting $\min_{u \in U} \dot{V}(x, u) = -L(x)$. The question remaining is "Does this provide optimal control?" The following section demonstrates that the answer is affirmative.

Before proceeding with the proof in the next section it should be pointed out that all of the approaches employing the Second Method yield closed-loop control. This is a feature that cannot be over-emphasized.

4.3 Optimality Theorem

In the preceding section, it was suggested that the selection of a V-function, $V(x)$, such that $\min_{u \in U} \dot{V}(x, u) = -L(x)$, would yield optimal control. In this section, a corresponding optimality theorem is stated and its proof given.

Before doing this, it is perhaps of value to state the basic optimization problem again. Given a linear system whose laws of motion can be described by

$$\dot{x} = \underline{A}x + \underline{B}u$$

it is desired to find an optimal, closed-loop, admissible control corresponding to a transition from x^0 to the origin with a cost functional of the form

$$J = \int_{t_0}^{t_1} L(x(t))dt$$

The control region, U , is the set of all control vectors, u , such that $\|u\|^2 \leq 1$.

For fixed values of x , $\dot{V}(x,u)$ becomes a continuous function of u . The minimum of this function with respect to all admissible controls is designated by $\dot{V}^0(x)$.

$$\dot{V}^0(x) = \min_{u \in U} \dot{V}(x,u) \quad (4.4)$$

Anticipating the results to follow, the corresponding minimizing control is denoted by u^0 .

Theorem 4.1 If there exists a Liapunov function, $V(x)$, with continuous second partial derivatives with respect to x and such that $\dot{V}^0(x) = -L(x)$, then the control u^0 which minimizes $\dot{V}(x,u)$ is an optimal control.

Before carrying out the proof of this theorem, consider the following lemma.

Lemma 4.1 If there exists a Liapunov function, $V(x)$, with continuous second partial derivatives with respect to x and such that $\dot{V}^0(x) = -L(x)$, then the gradient of $V(x)$, $\nabla V(x)$, satisfies the adjoint Equation (2.4).

The first step in the proof of the lemma is the minimization of $\dot{V}(x,u)$ as given by Equation (4.2) with respect to all admissible controls. The only term involving u is a scalar product of u and $\underline{B}'\nabla V(x)$. Thus by an argument similar to that presented in Section 2.4, u^0 is found to be

$$u^0 = \frac{-\underline{B}'\nabla V(x)}{\|\underline{B}'\nabla V(x)\|} \quad (4.5)$$

Substituting u^0 for u in Equation (4.2), one obtains

$$\dot{V}^0 = \nabla V'(x)\underline{A}x - \|\underline{B}'\nabla V(x)\| \quad (4.6)$$

Setting $\dot{V}^0(x) = -L(x)$ yields

$$\nabla V'(x)\underline{A}x - \|\underline{B}'\nabla V(x)\| = -L(x) \quad (4.7)$$

Now taking the partial derivative of both sides of Equation (4.7) with respect to x gives

$$\nabla(\nabla V'(x))\underline{A}x + \underline{A}'\nabla V(x) - \frac{\nabla(\nabla V'(x))\underline{B}\underline{B}'\nabla V(x)}{\|\underline{B}'\nabla V(x)\|} = -\nabla L(x)$$

Therefore

$$\nabla(\nabla V'(x))\underline{A}x = -\underline{A}'\nabla V(x) + \frac{\nabla(\nabla V'(x))\underline{B}\underline{B}'\nabla V(x)}{\|\underline{B}'\nabla V(x)\|} - \nabla L(x) \quad (4.8)$$

But from Equation (4.5)

$$u^0 = \frac{-\underline{B}'\nabla V(x)}{\|\underline{B}'\nabla V(x)\|}$$

and hence Equation (4.8) becomes

$$\nabla(\nabla V'(x))\underline{A}x = -\underline{A}'\nabla V(x) - \nabla(\nabla V'(x))\underline{B}u^0 - \nabla L(x) \quad (4.9)$$

Now consider the total time derivative of $\nabla V(x)$, again using the chain rule.

$$\begin{aligned}\frac{d}{dt}(\nabla V(x)) &= (\nabla(\nabla V'(x)))' \dot{x} \\ &= (\nabla(\nabla V'(x)))' \underline{A}x + (\nabla(\nabla V'(x)))' \underline{B}u^0\end{aligned}\quad (4.10)$$

By hypothesis $V(x)$ has continuous second partial derivatives and therefore the matrix $\nabla(\nabla V'(x))$ is symmetric. Thus $\nabla(\nabla V'(x)) = (\nabla(\nabla V'(x)))'$. Then substituting Equation (4.9) into Equation (4.10), one obtains

$$\frac{d}{dt}(\nabla V(x)) = -\underline{A}'\nabla V(x) - \nabla L(x)\quad (4.11)$$

Comparing Equation (4.11) with the adjoint Equation (2.13), one notes that $\nabla V(x)$ satisfies the adjoint equation, which completes the proof of the lemma.

Now returning to the proof of Theorem 4.1, $\nabla V(x)$ is substituted for p in the Hamiltonian as defined by Equation (2.9) to obtain

$$\begin{aligned}H(x, \nabla V(x), u) &= \nabla V'(x) \underline{A}x + \nabla V'(x) \underline{B}u + L(x) \\ &= \dot{V}(x, u) + L(x)\end{aligned}\quad (4.12)$$

Since $L(x)$ is not an explicit function of u ,

$$\min_{u \in U} H(x, \nabla V(x), u) = \min_{u \in U} \dot{V}(x, u) + L(x)$$

$$\text{Or} \quad H^0(x, \nabla V(x)) = \dot{V}^0(x) + L(x)\quad (4.13)$$

But by hypothesis, $\dot{V}^0(x) = -L(x)$ and hence

$$H^0(x, \nabla V(x)) = 0 \quad (4.14)$$

Therefore conditions one and two of the minimum principle have been satisfied and u^0 must be an optimal control, which completes the proof of Theorem 4.1. Again it should be noted that the control given by Equation (4.5) is a closed-loop control. In the next section, Theorem 4.1 is discussed further, in particular with respect to the classical Hamilton-Jacobi equation. The existence of Liapunov functions as required for this theorem is also discussed.

4.4 Hamilton-Jacobi Equation

It was demonstrated, in the previous section, that the optimal control problem with a constraint on the norm of the control vector is equivalent to the problem of solving the first-order partial differential equation

$$\dot{V}^0(x) = -L(x) \quad (4.15)$$

It is of interest to note that Equation (4.15) is, in fact, a special case of the classical Hamilton-Jacobi equation

$$H^0(x, \nabla V(x)) + \frac{\partial V(x)}{\partial t} = 0 \quad (4.16)$$

From the transversality conditions, one knows that $H^0(x, \nabla V(x)) = 0$ at the terminal time, t_1 . Since

$$\frac{dH^0(x, \nabla V(x))}{dt} = \frac{\partial H^0(x, \nabla V(x))}{\partial t}$$

and $H^0(x, \nabla V(x))$ does not contain t explicitly, $H^0(x, \nabla V(x))$ must be constant and identically zero for all t , $t_0 \leq t \leq t_1$. Therefore $\partial V(x)/\partial t$ is also identically zero and the Hamilton-Jacobi equation becomes

$$H^0(x, \nabla V(x)) = 0 \quad (4.17)$$

Thus for the problem of Section 2.3, one obtains

$$H^0(x, \nabla V) = \nabla V'(x) \underline{A}x - \| \underline{B}' \nabla V(x) \| + L(x) = 0$$

or

$$H^0(x, \nabla V(x)) = \dot{V}^0(x) + L(x) = 0$$

Use could have been made of this fact in the development of the previous section. However, it was felt that greater insight into the use of the Second Method was obtained by carrying out the proof in the manner presented. The knowledge that Equation (4.15) is the Hamilton-Jacobi equation does make it possible to conclude that the existence of a solution of Equation (4.15) is sufficient for optimal control to exist. This is an advantage over the minimum principle where only necessary conditions for optimality are given.

Next, one might ask if solutions of sufficient smoothness, i.e., continuous second partial derivatives, exist for Equation (4.15). Since the solutions of interest in the following chapter do, ipso facto, exist, the existence of solutions is not of prime importance here.

However, it is perhaps of interest to look briefly at the problem, even though a complete answer is not known.

First, it can be shown by example that if the control is scalar and the system is at least second-order, then there is no solution of sufficient smoothness. In fact, there is no solution with continuous first partial derivatives. On the other hand, Krassovskii (1959) has shown that if \underline{B} is non-singular and $L(x) = 1$, then a solution to Equation (4.15) exists with continuous partial derivatives of all order.

Hence, one is faced with a two-fold problem. First, a solution may not exist; and second, if one does exist, no general method of obtaining it is known. Therefore the basic optimization problem has not been solved. The necessary course of action is to obtain an approximate solution. In the next chapter, a method of modifying the Hamilton-Jacobi equation is followed by the presentation of a special class of solutions. From these solutions, a method is developed for designing effective sub-optimal control for a large, although limited, class of systems.

CHAPTER 5

EIGENVECTOR SCALAR PRODUCT SOLUTIONS

5.1 Introduction

In this chapter a special class of solutions of the Hamilton-Jacobi equation for the time-optimal problem is shown to exist. These solutions, called eigenvector scalar products, comprise the first of the three major contributions of this work. The second major contribution, which is contained in the last section of this chapter, is the development of a method for obtaining surfaces which bound the optimal isochrones from the outside. The next two chapters form the third major contribution, a method of designing effective sub-optimal control systems by the use of the eigenvector scalar product solutions.

The first part of this chapter presents a method of modifying the Hamilton-Jacobi equation in order to put the solution into a more convenient form. This is followed by the presentation of the eigenvector scalar product solutions. The last section of this chapter discusses the problem of bounding the optimum cost functional.

5.2 Modification of Hamilton-Jacobi Equation

A method of modifying the Hamilton-Jacobi equation is presented in this section which provides a more

convenient representation of the solutions to be discussed in the next section. One approach might be to make a non-linear transformation of state variables in order to reduce the Hamilton-Jacobi equation to some elementary form. To date this approach has not been very useful.

Another approach is to change to another Liapunov function $W(x)$, given by $G(V(x))$ where $V(x)$ is the optimum Liapunov function, i.e. a solution of Equation (4.15). In order for $W(x)$ to retain the basic nature of a Liapunov function, it is required that $G(V)$ satisfy the following conditions:

- 1) $G(V) > 0$ if $V > 0$
- 2) $G(0) = 0$
- 3) $dG(V)/dV > 0$ if $V > 0$
- 4) $\lim_{V \rightarrow \infty} G(V) = \infty$
- 5) $d^2G(V)/dV^2$ exists and is continuous

The effect that this transformation has on the Hamilton-Jacobi equation can be observed by considering the total time derivative of $W(x)$. Again \dot{W} is a function of both x and u and hence is written $\dot{W}(x,u)$.

$$\dot{W}(x,u) = \frac{dG(V)}{dV} \dot{V}(x,u) \quad (5.1)$$

Now minimizing $\dot{W}(x,u)$ with respect to all admissible controls, while remembering that $V(x)$ and hence $G(V(x))$ is not a function of u , yields

$$\begin{aligned}
\min_{u \in U} \dot{W}(x, u) &= \frac{dG(V)}{dV} \min_{u \in U} \dot{V}(x, u) \\
&= \frac{dG(V)}{dV} \dot{V}^0(x)
\end{aligned} \tag{5.2}$$

The minimum of $\dot{W}(x, u)$ with respect to $u \in U$ is designated by $\dot{W}^0(x)$. Then Equation (5.2) becomes

$$\dot{W}^0(x) = \frac{dG(V)}{dV} \dot{V}^0(x) \tag{5.3}$$

But, by assumption, $V(x)$ is a solution of Equation (4.15) and hence $\dot{V}^0(x) = -L(x)$. Therefore Equation (5.3) becomes

$$\dot{W}^0(x) = -L(x) \frac{dG(V)}{dV} \tag{5.4}$$

Since $dG(V)/dV$ is positive for V greater than zero, G must be monotone increasing on the interval $[0, \infty)$. Then according to conditions 1) and 2) above, G must map the interval $[0, \infty)$ onto the interval $[0, \infty)$ in a one-to-one fashion. Therefore G possesses a unique inverse function I on the interval $[0, \infty)$. Since both $V(x)$ and $W(x)$ are required to be positive definite, this is the only region of interest. Therefore

$$V(x) = I(W(x)) \tag{5.5}$$

Then substituting for $V(x)$ in Equation (5.4) gives

$$\dot{W}^0(x) = -L(x) \frac{dG(I(W(x)))}{dV} \tag{5.6}$$

Now letting $F(W) = \frac{dG(I(W))}{dV}$, Equation (5.6) becomes

$$\dot{W}^0(x) = -L(x)F(W(x)) \quad (5.7)$$

This equation is called the modified Hamilton-Jacobi equation. For the case of time optimal control, $L(x) = 1$, and Equation (5.7) reduces to

$$\dot{W}^0(x) = -F(W(x)) \quad (5.8)$$

By combining the results of this section with the theorem of Section 4.3, the following optimality theorem results.

Theorem 5.1 If there exists a Liapunov function, $W(x)$, with continuous second partial derivatives, such that $\dot{W}^0(x) = -L(x)F(W(x))$ where $F(W) = dG(I(W))/dV$ and G satisfies the conditions given above, then the control, u^0 , which minimizes $\dot{W}(x, u)$, is an optimal control.

The first step in the proof of this theorem is to obtain the Liapunov function, $V(x)$, which corresponds to $W(x)$. Substituting $W = G(V)$ into the definition of $F(W)$ yields

$$F(G(V)) = dG(I(G(V)))/dV \quad (5.9)$$

However I is the inverse of G and hence $I(G(V)) = V$; then Equation (5.9) becomes

$$F(G(V)) = dG(V)/dV \quad (5.10)$$

By antidifferentiation $G(V)$ can be obtained from Equation (5.10). By hypothesis this $G(V)$ must satisfy the conditions given above. Hence $V(x)$ given by $I(W)$ must be a Liapunov function if $W(x)$ is. Condition 5) on $G(V)$ assures that if $W(x)$ has continuous second partial derivatives, then $V(x)$ does also. Thus the first portion of Theorem 4.1 has been satisfied.

Next consider $\dot{V}(x,u)$ which may be obtained as

$$\dot{V}(x,u) = \frac{dV(x)}{dW} \dot{W}(x,u) \quad (5.11)$$

Since neither $W(x)$ nor $V(x)$ are functions of u , the same control u^0 must minimize both $\dot{V}(x,u)$ and $\dot{W}(x,u)$ and hence Equation (5.11) becomes

$$\dot{V}^0(x) = \frac{dV(x)}{dW} \dot{W}^0(x)$$

By hypothesis $\dot{W}^0(x) = -L(x)F(W(x))$ and therefore one obtains

$$\dot{V}^0(x) = -L(x) [dV/dW] [dG(V)/dV] \quad (5.12)$$

But $G(V) = W$ and hence

$$\begin{aligned} [dV/dW] [dG(V)/dV] &= [dV/dW] [dW/dV] \\ &= 1 \end{aligned}$$

Therefore Equation (5.12) becomes

$$\dot{V}^0(x) = -L(x)$$

Hence $V(x)$ satisfies the conditions of the Theorem 4.1 and u^0 must be an optimal control which completes the proof of the theorem.

For the minimum time problem, this theorem becomes
Theorem 5.2 If there exists a Liapunov function $W(x)$, with continuous second partial derivatives, such that $\dot{W}^0(x) = F(W(x))$, then u^0 is a time-optimal control.

This last theorem embodies the basic concept of the method presented by Nahi (1964) for obtaining time-optimal control by the use of the Second Method. However, by the development presented here, greater insight and information are gained with regard to the function F .

Hence by the use of the transformation G , the problem of finding a solution for the Hamilton-Jacobi equation has been changed to the problem of finding a solution for the modified Hamilton-Jacobi equation. By means of such a transformation, it is hoped that the solution can be facilitated. This procedure is, in fact, a special case of the procedure of canonic transformations used in classical mechanics and partial differential equation theory.

One transformation, G , which is of particular importance in the next section is

$$W = G(V) = \left[\frac{K_2}{K_1} \left(\exp \left(\frac{K_1 V}{2} \right) - 1 \right) \right]^2 \quad (5.13)$$

Then the inverse of G is given by

$$V = I(W) = \frac{2}{K_1} \ln\left(\frac{K_1\sqrt{W}}{K_2} + 1\right) \quad (5.14)$$

Therefore Equation (5.7) becomes

$$\dot{W}^0(x) = -L(x) \left[K_1 W(x) + K_2 \sqrt{W(x)} \right] \quad (5.15)$$

For the time-optimal case, one obtains

$$\dot{W}^0(x) = -K_1 W(x) - K_2 \sqrt{W(x)} \quad (5.16)$$

This equation plays an important role in the next section.

5.3 Eigenvector Scalar Products

In this section a particular class of solutions of the Hamilton-Jacobi equation is developed. Because of the manner in which these solutions are formed, they are called eigenvector scalar product solutions. For the material to be presented in the remaining portion of this chapter and in the next chapter, two additional assumptions are added to the basic optimization problem as formulated in Section 2.3. First, only time-optimal control is considered, i.e. $L(x) = 1$. Second, the eigenvalues of the matrix A in Equation (2.2) must be real, non-positive, and distinct.

In the preceding section, it was shown that time optimal control could be obtained by finding a Liapunov function, $W(x)$, such that $\dot{W}^0(x) = -F(W(x))$. The following theorem, due to Malkin (1958), establishes a necessary and

sufficient condition for $\dot{W}(x) = \lambda W(x)$ for uncontrolled linear systems.

Theorem 5.3 For systems whose laws of motion are of the form $\dot{x} = Ax$ there exist Liapunov functions such that $\dot{W}(x) = \lambda W(x)$ if and only if $\lambda = m_1 \lambda_1 + m_2 \lambda_2 + \dots + m_n \lambda_n$ and $W(x)$ is given by

$$W(x) = (q^1, x)^{m_1} (q^2, x)^{m_2} \dots (q^n, x)^{m_n}$$

where the λ_i 's are the eigenvalues of A and q_i is the eigenvector of A' associated with λ_i and the m_i 's are positive integers.

The reader is referred to Malkin (1958) for a proof of the necessity portion of the above theorem, which is somewhat involved and not of particular importance for the present discussion. The proof of the sufficiency of the above theorem is presented below, since it is useful in the following work. However, before beginning this proof, consider the following lemma.

Lemma 5.1 If q is an eigenvector of A' and λ is the associated eigenvalue, and if $W(x) = q'x$, then

$$\dot{W}(x) = \lambda W(x)$$

For $W(x) = q'x$ $\dot{W}(x)$ is given by

$$\dot{W}(x) = q'\dot{x} = q'Ax \quad (5.17)$$

But q is an eigenvector of \underline{A}' , hence

$$\underline{A}'q = \lambda q \quad (5.18)$$

Or, taking the transpose of both sides of Equation (5.18), one obtains

$$q' \underline{A} = \lambda q' \quad (5.19)$$

Substituting Equation (5.19) into Equation (5.17) yields

$$\dot{W}(x) = \lambda q'x = \lambda W(x)$$

and the proof of the lemma is completed.

Returning to the proof of the theorem, consider a Liapunov function of the form

$$W(x) = (q^1'x)^{m_1} (q^2'x)^{m_2} \dots (q^n'x)^{m_n} \quad (5.20)$$

Now let $W_1(x) = q^1'x$ and then Equation (5.20) becomes

$$W(x) = W_1(x)^{m_1} W_2(x)^{m_2} \dots W_n(x)^{m_n}$$

Now taking the total time derivative of $W(x)$, one obtains

$$\dot{W}(x) = m_1 W_1^{m_1-1} \dot{W}_1 W_2^{m_2} \dots W_n^{m_n} + \dots + m_n W_1^{m_1} \dots W_n^{m_n-1} \dot{W}_n \quad (5.21)$$

But from Lemma 5.1, $\dot{W}_1 = \lambda_1 W_1$, then Equation (5.21) becomes

$$\begin{aligned} W(x) &= \lambda_1 m_1 W_1^{m_1-1} W_2^{m_2} \dots W_n^{m_n} + \dots + \lambda_n m_n W_1^{m_1} \dots W_n^{m_n-1} \\ &= (\lambda_1 m_1 + \lambda_2 m_2 + \dots + \lambda_n m_n) W(x) \end{aligned}$$

This completes the proof of the theorem.

Since $\dot{W}(x)$ turns out to be a function of $W(x)$, one is led to consider a Liapunov function of the form of Equation (5.20) as a possible solution of the modified Hamilton-Jacobi equation. The following theorem indicates that there are, in fact, solutions of this form.

Theorem 5.4 If q is an eigenvector of A' and λ is the associated eigenvalue, then $W(x) = (q'x)^2$ is a solution of the modified Hamilton-Jacobi Equation (5.16), i.e. $\dot{W}^0(x) = -K_1 W(x) - K_2 \sqrt{W(x)}$ where $K_1 = -2\lambda$ and $K_2 = 2\|B'q\|$.

As a first step in the proof, consider the following lemma.

Lemma 5.2 For any matrix P such that $P = pp'$ and any matrix B , $P'BB'P = \|B'p\|^2 P$.

Writing out $P'BB'P$ in full, one obtains

$$P'BB'P = pp'BB'pp'$$

Now consider the $p'BB'p$ portion of this expression. B is an n by r matrix, while p is an n by 1 column matrix (vector). Hence the product $p'B$ is an 1 by r matrix, and $B'p$ is an r by 1 matrix. Therefore the product $p'BB'p$ must be an 1 by 1 matrix, or a scalar, whose value is $\|B'p\|^2$. Therefore

$$\begin{aligned} P'BB'P &= p(\|B'p\|^2)p' \\ &= \|B'p\|^2 P \end{aligned} \quad (5.22)$$

which completes the proof of the lemma. It should be pointed out that B is not required to be non-singular.

The next step, in the proof of the theorem, is to rewrite $W(x)$ in a new form. Since $q'x = x'q$, then $W(x) = q'xq'x$ can be written as

$$\begin{aligned} W(x) &= x'qq'x \\ &= x'\underline{Q}x \end{aligned} \quad (5.23)$$

where $\underline{Q} = qq'$. It should be noted that \underline{Q} is positive semi-definite and symmetric. Now taking the gradient of $W(x)$, one obtains

$$\nabla W(x) = 2\underline{Q}x \quad (5.24)$$

By substituting $W(x)$ for $V(x)$ in Equation (4.6), $\dot{W}^0(x)$ is given by

$$\dot{W}^0(x) = \nabla W'(x)\underline{A}x - \|\underline{B}'\nabla W(x)\| \quad (5.25)$$

Substituting Equation (5.24) into Equation (5.25) and expanding $\|\underline{B}'\nabla W(x)\|$, one obtains

$$\begin{aligned} \dot{W}^0(x) &= 2x'\underline{Q}'\underline{A}x - 2\sqrt{x'\underline{Q}'\underline{B}\underline{B}'\underline{Q}x} \\ &= 2x'qq'\underline{A}x - 2\sqrt{x'\underline{Q}'\underline{B}\underline{B}'\underline{Q}x} \end{aligned} \quad (5.26)$$

But q is an eigenvector of \underline{A}' and hence $q'\underline{A} = \lambda q'$. From the lemma above, $\underline{Q}'\underline{B}\underline{B}'\underline{Q} = \|\underline{B}'q\|^2\underline{Q}$. Therefore Equation (5.26) becomes

$$\dot{W}^0(x) = 2\lambda(x'\underline{Q}x) - 2\|\underline{B}'q\|\sqrt{x'\underline{Q}x}$$

or

$$\dot{W}^0(x) = 2\lambda W(x) - 2\|\underline{B}'q\|\sqrt{W(x)} \quad (5.27)$$

Hence $W(x) = (q'x)^2$ satisfies Equation (5.16) and the proof of the theorem is completed. Solutions of this type are called eigenvector scalar product solutions since they are scalar products of eigenvectors with the state vector.

By the use of Equation (5.14) the Liapunov function, $V(x)$, which is a solution to the Hamilton-Jacobi Equation (4.15) is given by

$$V(x) = \frac{1}{-\lambda} \ln \left(\frac{-\lambda |q'x|}{\|B'q\|} + 1 \right) \quad (5.28)$$

It can be easily verified by direct substitution that $\dot{V}^0(x) = 1$. The corresponding optimal control is given by

$$u^0(x) = \frac{-B'qq'x}{\|B'qq'x\|} \quad (5.29)$$

The obvious simplicity of the form of $W(x)$ as compared to $V(x)$ points out the reason for the use of the modified Hamilton-Jacobi equation. However, $V(x)$ is also important, since surfaces of constant $V(x)$ are surfaces of constant time. This point is discussed further in the next section, which is concerned with bounding the optimum transition time. Before proceeding to the next section, it is perhaps wise to consider a particular example of the solutions presented above.

Example 5.1 The equations of motion of the system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (5.30)$$

It is desired to find the eigenvector scalar product solutions for this problem and to show that they satisfy the modified Hamilton-Jacobi equation. The corresponding solutions of the Hamilton-Jacobi equation are also to be found and verified.

By standard methods the eigenvalues are found to be -1, -2 with the corresponding (unnormalized) eigenvectors of \underline{A}' being (2,1) and (1,1). It should be noted that any other set of eigenvectors of \underline{A}' could have been chosen, since the resulting optimal control and Liapunov function is unchanged. The above set was chosen for its computational convenience.

There are two solutions of the modified Hamilton-Jacobi equation which can be obtained by the above method, corresponding to the two eigenvectors.

First, for the eigenvalue -1, one obtains

$$W_1(x) = (q^{1'}x)^2 = (2x_1 + x_2)^2 \quad (5.31)$$

and $\dot{W}_1^0(x)$ is given by

$$\dot{W}_1^0(x) = -2W_1(x) - 2\sqrt{W_1(x)}$$

The corresponding solution of the Hamilton-Jacobi equation is

$$V_1(x) = \ln(|2x_1 + x_2| + 1) \quad (5.32)$$

while the optimal control as given by Equation (5.29) is

$$u^0(x) = \frac{-(2x_1 + x_2)}{|2x_1 + x_2|} \quad (5.33)$$

The total time derivative of $V_1(x)$ is then given by

$$\dot{V}_1(x,u) = \left[\frac{1}{|2x_1 + x_2| + 1} \right] \left[\frac{(2x_1 + x_2)}{|2x_1 + x_2|} \right] [2\dot{x}_1 + \dot{x}_2]$$

Now substituting from Equation (5.30) one obtains

$$\dot{V}_1(x,u) = \left[\frac{1}{|2x_1 + x_2| + 1} \right] \left[\frac{(2x_1 + x_2)}{|2x_1 + x_2|} \right] [-2x_1 - x_2 + u]$$

If $u^0(x)$ as given by Equation (5.33) is now substituted for u , $\dot{V}_1(x,u)$ becomes $\dot{V}_1^0(x)$.

$$\begin{aligned} \dot{V}_1^0(x) &= \left[\frac{(2x_1 + x_2)}{|2x_1 + x_2|^2 + |2x_1 + x_2|} \right] \left[-(2x_1 + x_2) - \frac{(2x_1 + x_2)}{|2x_1 + x_2|} \right] \\ &= -1 \end{aligned}$$

Hence $V_1(x)$ satisfies the Hamilton-Jacobi equation as predicted. Then for the second eigenvalue, one obtains

$$W_2(x) = (q^{2'}x)^2 = (x_1 + x_2)^2 \quad (5.34)$$

and $V_2(x)$ is given by

$$V_2(x) = \frac{1}{2} \ln(2|x_1 + x_2| + 1) \quad (5.35)$$

Again it can be readily verified that $V_2(x)$ satisfies the Hamilton-Jacobi equation. These results are used again in Chapter 7.

Hence it has been verified that the scalar functions given by Equation (5.28) are solutions of the Hamilton-Jacobi equation. However, these solutions cannot be used

directly, since the functions are only positive semi-definite. In the next two chapters, methods of employing these solutions to obtain sub-optimal control are developed. However, before doing this it is perhaps of value to examine the eigenvector scalar product solutions in more detail. In the next section these are investigated with respect to bounding the optimum cost functional. In this manner it is hoped that the reader is able to acquire a better understanding of the eigenvector scalar product solutions.

5.4 Bounds on Transition Time

In Section 4.2 it was briefly mentioned that if $\dot{V}(x) = -L(x)$, then surfaces of constant $V(x)$ become surfaces of constant cost. This point perhaps needs further elaboration. In the case of time optimal control, $\dot{V}(x) = -1$, and hence integrating with respect to t from t_0 to t_1 , one obtains

$$V(x^1) - V(x^0) = t_0 - t_1 \quad (5.36)$$

If the terminal state is taken to be the origin, then $V(x^1) = 0$, and

$$t_1 - t_0 = V(x^0)$$

Thus the value of the Liapunov function at the initial state of the system is equal to the transition time. If a Liapunov function $V_0(x)$ has been found such that $\dot{V}_0(x) = -1$, then $V_0(x^0)$ is equal to the minimum transition time

from x^0 to the origin. Let S_0 be the surface composed of all points x such that $V_0(x) = T_0$ where $V_0(x)$ is the solution of the Hamilton-Jacobi equation, the optimum Liapunov function. Then S_0 is the set of all points from which it is possible to reach the origin in a transition time T_0 by the use of time optimal control. This surface must be smooth and enclose the origin. Figure 5.1 shows a two dimensional example where the surface S_0 has become the closed curve designated by S_0 . Such a surface is called an isochrone. The problem of finding optimal control is actually a problem of finding the equation for the isochrone or $V(x)$.

Since it is normally impossible to obtain the exact solution of the Hamilton-Jacobi equation, it is necessary to approximate the solution. If such an approximate solution, $V_a(x)$, is found, then let S_1 be the surface composed of all points x such that $V_a(x) = T_0$, i.e. the set of all points from which the origin can be reached in T_0 seconds by the use of sub-optimal control. The surface S_1 must be within or at most tangent to S_0 as shown in Figure 5.1.

One method for judging the quality of a sub-optimal control is now obvious. The more nearly the surface S_1 coincides with the surface S_0 , the better the sub-optimal control. However, since the surface S_0 is generally not known, such a method of judging the quality of the approximation is rather academic. Some other method is needed.

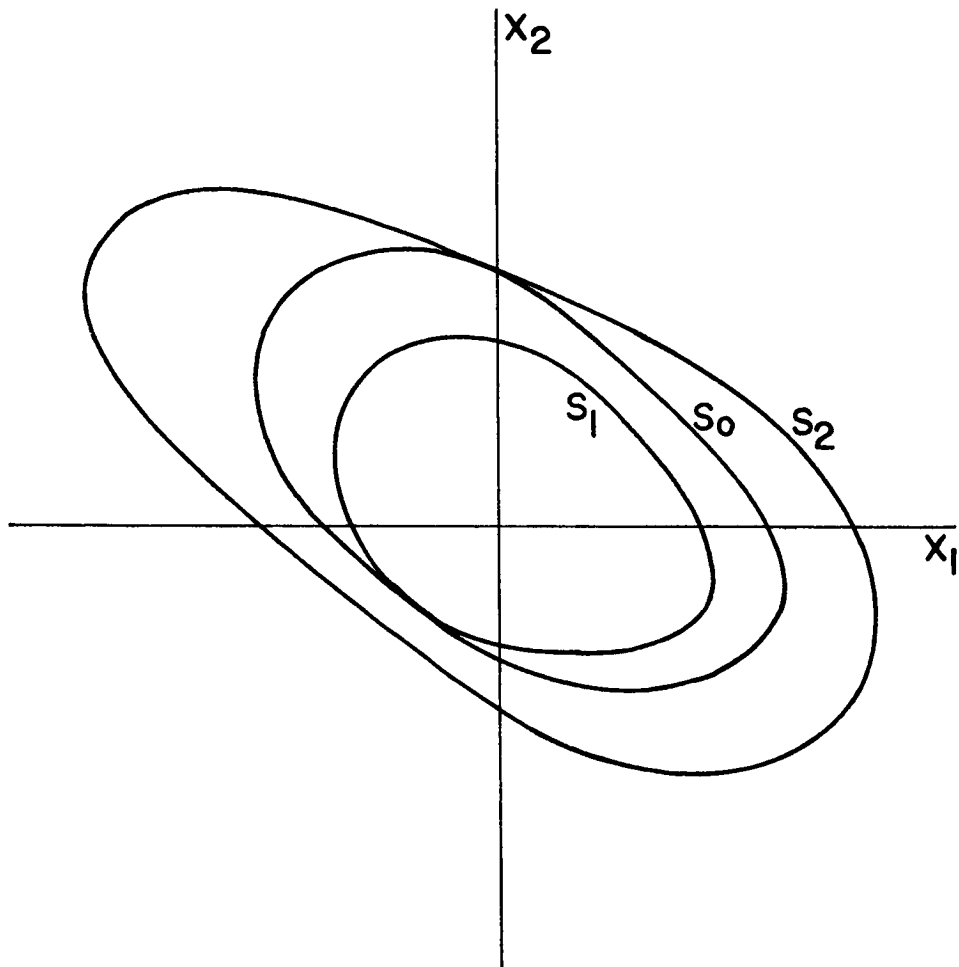


Figure 5.1 Typical Isochrones

One such method is to find another surface S_2 which is entirely outside or at most tangent to S_0 , as shown in Figure 5.1. If such a surface could be found in a relatively easy and straight-forward manner, the quality of an approximation could be determined in the following manner. If S_1 and S_2 were close, then S_1 must be a good approximation, since S_1 must be at least as close to S_0 as it is to S_2 . However, if S_1 and S_2 were far apart, no conclusion could be reached regarding the quality of the control, since there would be no knowledge with respect to the relation of S_1 and S_0 . This situation should be compared with the basic concept of the Second Method, where a failure to construct a Liapunov function generally yields no concrete results with respect to stability.

It should be noted that the surface S_2 does, in general, not correspond to any physical control situation. If there did exist an admissible control which would take the system from S_2 to the origin in T_0 seconds, this would contradict the assumption that S_0 was optimal. However, there may be points on S_2 which correspond to points on S_0 , and hence from these points the system can be returned to the origin in T_0 seconds.

The eigenvector scalar product solutions, as developed in the preceding section, provide an unusually simple method for obtaining a S_2 -type surface. Although the

surface generated does not uniformly approximate S_0 from the outside, it is tangent to S_0 at several points, as is pointed out later.

Consider for a moment the interpretation that one may give to Liapunov functions which are given by Equation (5.28).

$$V(x) = \frac{1}{-\lambda} \ln \left(\frac{-\lambda |q'x|}{\|B'q\|} + 1 \right) \quad (5.28)$$

In this case $V(x^1)$ is zero if and only if $q'x^1$ is zero. Thus the value of $V(x^0)$ does not correspond to the minimum transition time from x^0 to the origin, but rather from x^0 to the hyperplane defined by $q'x^1 = 0$. Since the surface $V(x) = T_0$ corresponds to the surface $|q'x| = K$, a constant, which is two hyperplanes, $V(x) = T_0$ is actually two hyperplanes symmetrically placed about $q'x = 0$. See Figure 5.2 for a two dimensional example of these $V(x)$ equals a constant surfaces and typical trajectories of the system.

Since the origin is one point on the hyperplane $V(x) = 0$, then $V(x^0)$ must be equal to or less than the minimum transition time from x^0 to the origin. If it were greater, there would exist a control which would transfer the system to the hyperplane in a time less than $V(x^0)$, which contradicts the optimality of the Liapunov function given by Equation (5.28). Therefore the surface (hyperplanes) $V(x) = T_0$ must be entirely outside or at most tangent to the S_0 surface.

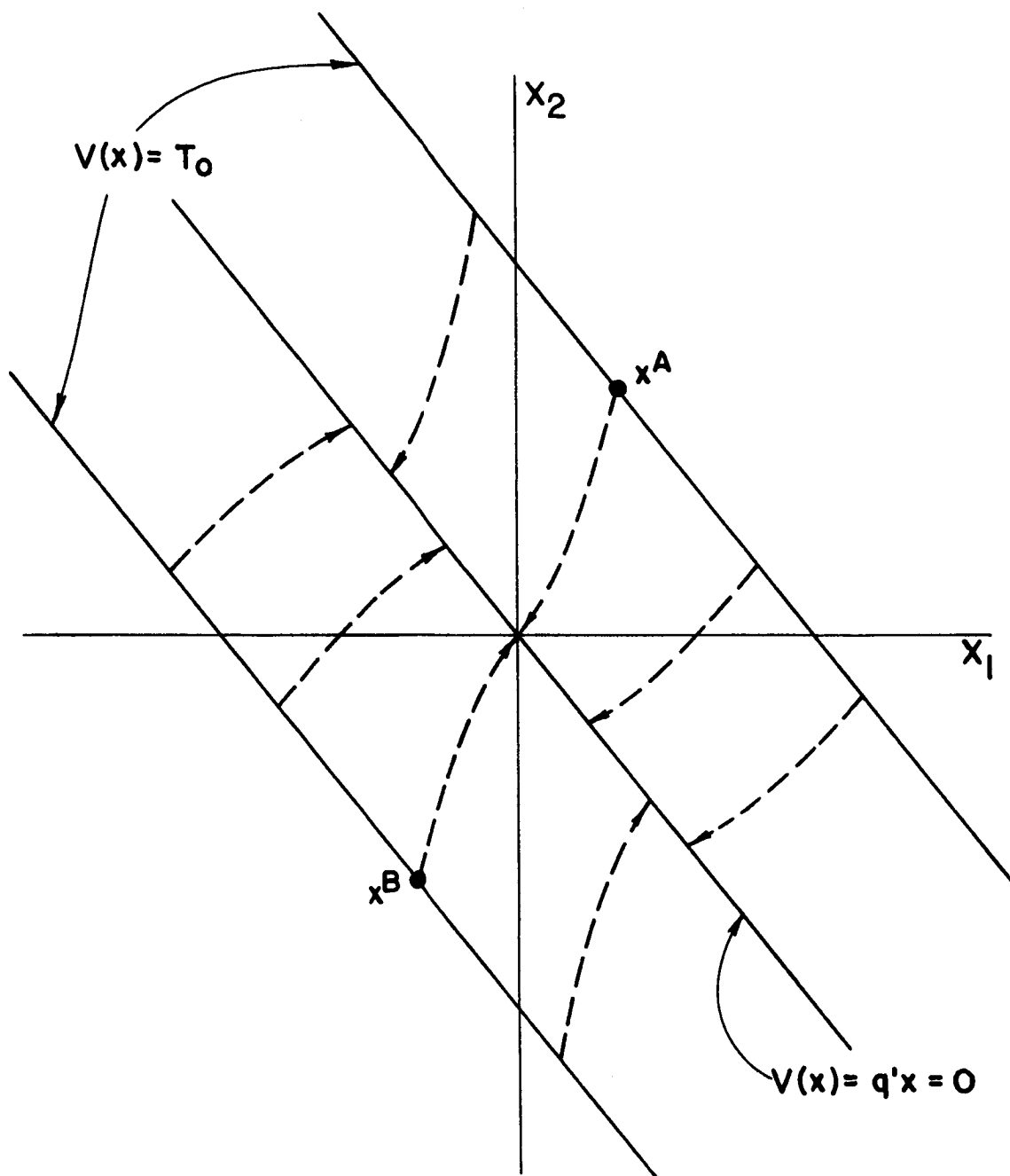


Figure 5.2 Surfaces of Constant $V(x)$ for Eigenvector Scalar Product Solutions

It is very simple to show that the $V(x) = T_0$ surface must be tangent to S_0 in two places. Since the system is controllable, there must be two points (one on each hyperplane) from which the origin is reached in T_0 seconds as a special case of reaching the hyperplane $q'x = 0$. See points x^A and x^B in Figure 5.2. But these points must be on S_0 ; otherwise they would contradict the optimality of S_0 . Hence there are two points at which the $V(x) = T_0$ surface is tangent to S_0 .

Since the n eigenvalues are distinct, the eigenvectors are linearly independent and hence the n surfaces (hyperplanes) are non-coplanar (should probably be non-cohyperplanar). Therefore the boundary of the set of points for which $V_i(x) \leq T_0$, $i = 1, 2, \dots, n$ is a closed surface. See Figure 5.3 where the cross-hatched area is such a set. However, every point on this surface must be outside or on the S_0 surface, since each boundary point is on some surface $V_i(x) = T_0$, and by the argument above, each such point is outside or on S_0 . Therefore this surface must be an S_2 -type surface.

The fact that there are $2n$ points at which the above S_2 surface is tangent to the S_0 surface can be argued in the following manner. By the argument presented above, the surface $V_1(x) = T_0$ must be tangent to S_0 at two points. Since from these points it is possible to reach the origin in T_0 seconds, it is also possible to reach all of the other

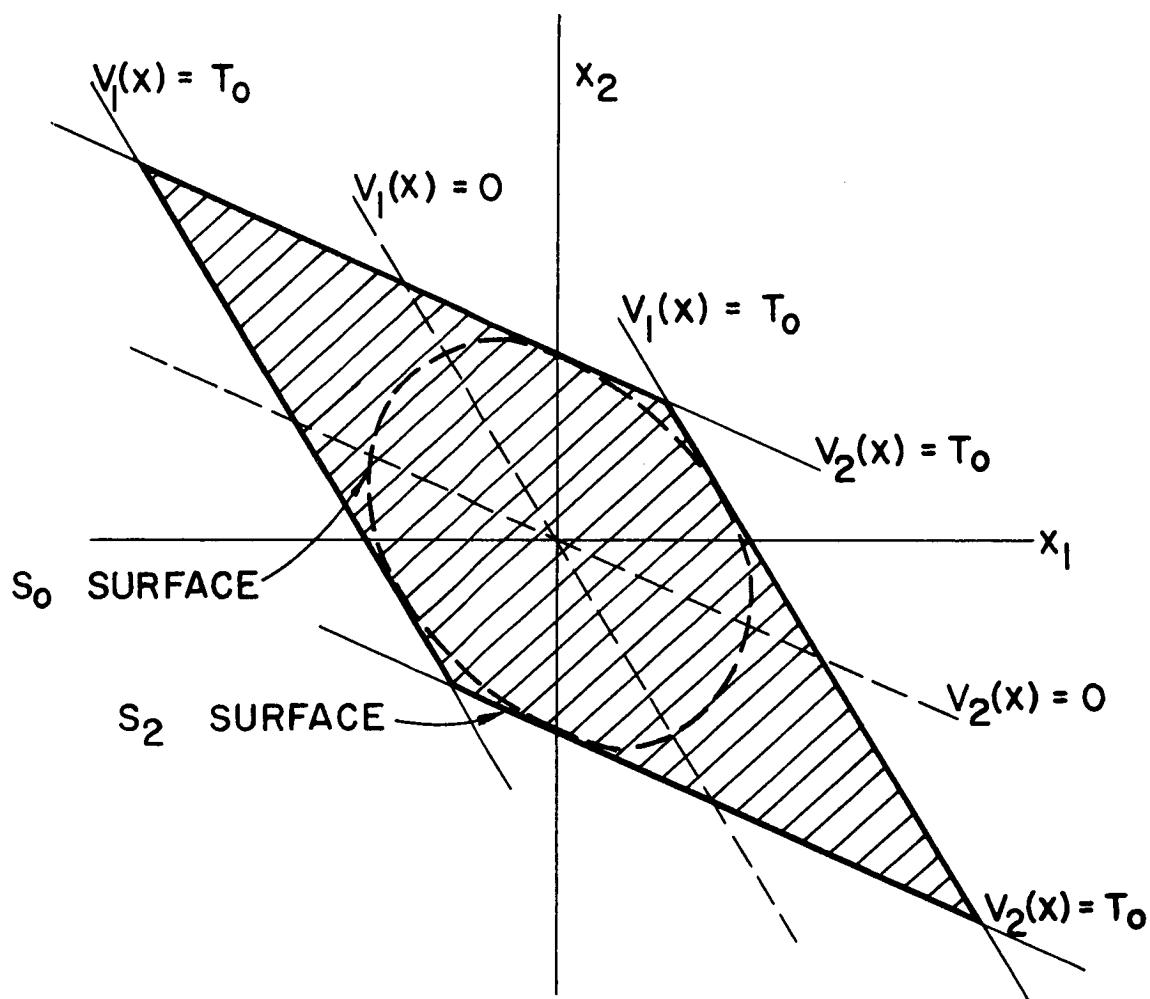


Figure 5.3 A Typical S_2 Surface Bounding
The Optimal S_0 Surface

$V_1(x) = 0$ surfaces in T_0 seconds from these points. Therefore these points must be on both the S_2 and the S_0 surfaces. By a similar argument, it can be concluded that there are $2n$ points which are common to S_2 and S_0 . See Figure 5.3 which shows a typical S_2 surface generated by this method.

Thus by the use of the eigenvector scalar product solutions it is possible to obtain a relatively good S_2 surface in an unusually simple manner. The inability to find such surfaces has been a serious difficulty in designing approximately time-optimal systems in the past. Without such S_2 surfaces, it is impossible to judge the quality of a sub-optimal system without actually obtaining the optimal solution. These S_2 surfaces are used in the next chapter in order to evaluate the sub-optimal method presented there.

CHAPTER 6

SUB-OPTIMAL CONTROL WITH NON-SINGULAR CONTROL MATRIX

6.1 Introduction

In this chapter a method for designing sub-optimal control systems is developed, based on the eigenvector scalar product solutions presented in the previous chapter. The control matrix, \underline{B} , is assumed to be non-singular for the work presented in this chapter. The method is developed first for second-order systems in order to be able to carry out a geometric representation and interpretation of the method. A second-order example completes the presentation.

Following the development of the sub-optimal control method for second-order systems, a generalization to n-th order systems is made. A third-order example is used to illustrate the generalization. The chapter concludes with a brief discussion of the method and its application.

It is perhaps of value to state the basic optimization problem that is considered in this chapter. For linear systems whose laws of motion are described by

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}u$$

where the eigenvalues of \underline{A} are real, distinct and non-positive and the matrix \underline{B} is non-singular, it is desired to find time-optimal, closed-loop control corresponding to a

transition from x^0 to the origin. The control region, U , is the set of all control vectors, u , such that $\|u\|^2 \leq 1$.

6.2 Sub-Optimal Control of Second-Order Systems

In this section, a method of sub-optimal control of second-order systems with non-singular control matrices is developed. Before beginning this development, it is necessary to modify slightly the eigenvector scalar product solution of the previous chapter.

This modification involves a generalization of the bound on the norm of the control vector from unity to some unspecified constant, ρ . If such a change is made either by transforming the control vector or by repeating the work of Chapter 5, the Liapunov function, $V(x)$, as given by Equation (5.28) becomes

$$V(x) = \frac{1}{-\lambda} \ln \left(\frac{-\lambda |q'x|}{\rho \|B'q\|} + 1 \right) \quad (6.1)$$

and the corresponding optimal control is

$$u^0(x) = \frac{-\rho B'qq'x}{\|B'qq'x\|} \quad (6.2)$$

As would be expected, for a fixed initial condition, x^0 , increasing ρ causes $V(x^0)$ to decrease, i.e. the transition time decreases with increasing control effort. Since the numerical value of $V(x)$ is dependent on both the system's state and on the norm of the control vector, $V(x)$ is written

as $V(x, \rho)$ to indicate this relation. Similarly, u is written as $u(x, \rho)$.

A general second-order system with real, non-positive and distinct eigenvalues, λ_1 and λ_2 is considered in this section. For each eigenvalue, there is an eigenvector, designated by q^1 and q^2 respectively. Associated with the two eigenvectors are two Liapunov functions given by Equation (6.1), $V_1(x, \rho)$ and $V_2(x, \rho)$, and their corresponding optimal controls $u^1(x, \rho)$ and $u^2(x, \rho)$.

It may seem strange to speak of two optimal controls. The reader, however, is reminded of the significance of these controls. The control u^1 transfers the system from any initial point to the surface $V_1 = 0$ in minimum time, while u^2 transfers the system to the surface $V_2 = 0$ in minimum time. Hence it is possible to have two optimal controls since the problem is different in each case.

For some point in the state space, x^0 , the control given $u^1(x, \rho)$ transfers the system from x^0 to some point, x^1 , on the surface $V_1(x, \rho) = 0$ in minimum time. Typical points and the corresponding optimal trajectory are shown in Figure 6.1. In the case of second-order systems, surfaces of $V(x)$ equals a constant become lines. That $u^1(x, \rho)$ is, in fact, constant during this transition can be shown in the following manner.

For a given eigenvector, q^1 , $B'q^1$ in Equation (6.2) is a constant vector while $q^{1'}x$ is a scalar. Hence $u^1(x, \rho)$

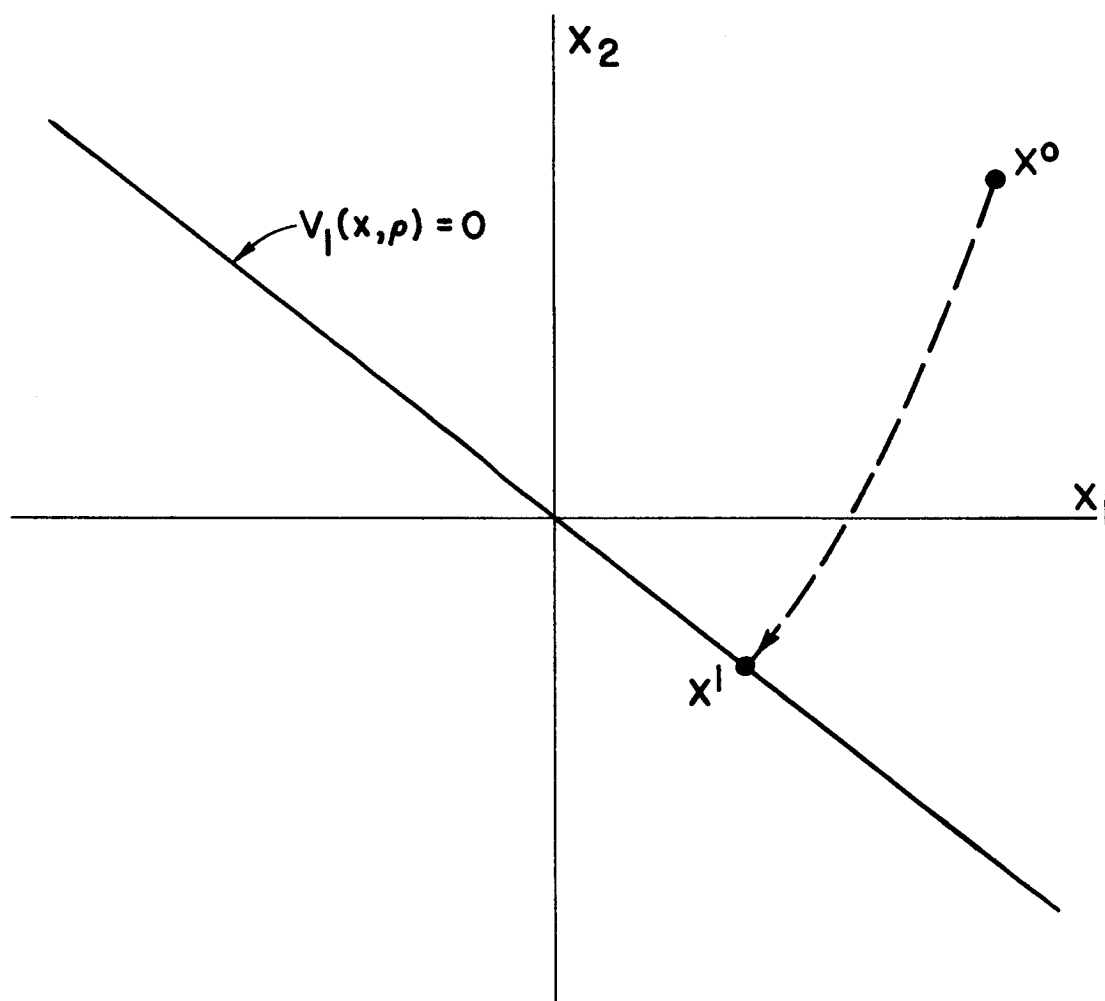


Figure 6.1 Typical Eigenvector Scalar Product Optimal Trajectory

must be a vector parallel to $\underline{B}'q^1$; its direction is determined by the sign of the scalar quantity $q^1'x$. Since the $x_1(t)$'s are continuous functions of time, it is necessary for $q^1'x$ to be zero before it can change sign. But if $q^1'x$ is zero, then $V_1(x, \rho)$ is also zero. Therefore the sign of $q^1'x$ cannot change during the transition from x^0 to x^1 . Hence $u^1(x, \rho)$ must be a constant vector, whose norm is equal to ρ and whose direction is given by $\underline{B}'q^1q^1'x$. Figure 6.2 shows a typical control vector, $u^1(x, \rho)$.

Consider now another constant control vector, u , as shown in Figure 6.2 which is equal to the addition of $u^1(x, \rho)$ and any arbitrary constant vector r perpendicular to $u^1(x, \rho)$. Therefore

$$u = u^1(x, \rho) + r \quad (6.3)$$

where r is any constant vector such that $r'u^1(x, \rho) = 0$. The transition time from the point x^0 to the line $V_1(x, \rho) = 0$ is independent of r ; this fact can be shown in the following manner.

Consider the Liapunov function $V_1(x, \rho)$ as given by Equation (6.1). Now computing its total time derivative, one obtains

$$\begin{aligned} \dot{V}_1(x, \rho) &= \frac{x'qq'\dot{x}}{-\lambda|q'x|^2 + \rho\|\underline{B}'qq'x\|} \\ &= \frac{x'qq'(Ax + Bu)}{-\lambda|q'x|^2 + \rho\|\underline{B}'qq'x\|} \end{aligned} \quad (6.4)$$

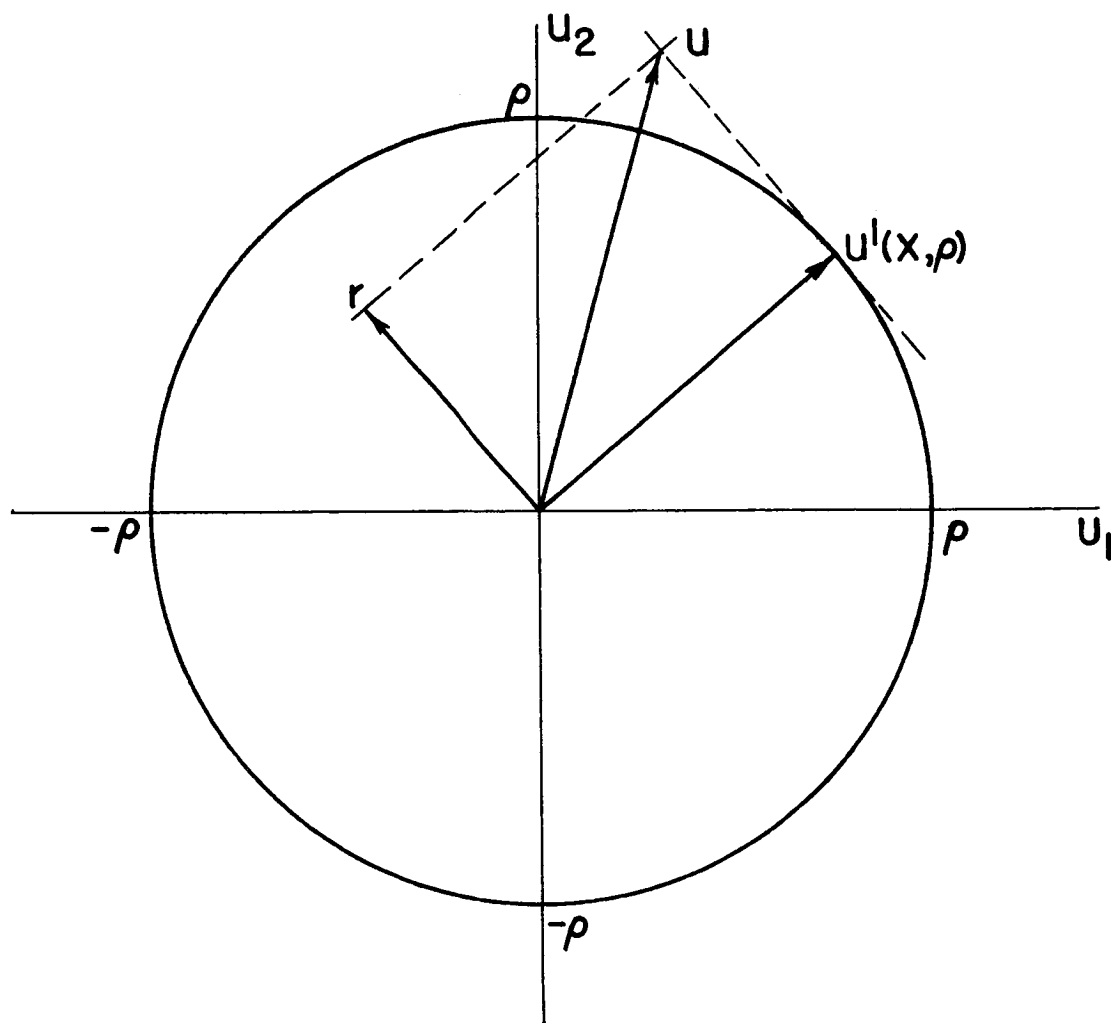


Figure 6.2 Typical Optimal and Non-Optimal Controls

Substituting Equation (6.3) for u gives

$$\dot{V}_1(x, \rho) = \frac{x'qq'Ax + x'qq'Bu^1(x, \rho) + x'qq'Br}{-\lambda|q'x|^2 + \rho\|B'qq'x\|} \quad (6.5)$$

However, r is perpendicular to $u^1(x, \rho)$ and by the argument above $u^1(x, \rho)$ is parallel to $B'q$. Therefore r must be perpendicular to $B'q$ and the scalar product of r and $B'q$ must be zero, i.e. $q'Br = 0$. Hence the third term in the numerator of Equation (6.5) must be zero. Therefore

$$\dot{V}_1(x, \rho) = \frac{x'qq'Ax + x'qq'Bu^1(x, \rho)}{-\lambda|q'x|^2 + \rho\|B'qq'x\|} \quad (6.6)$$

By direct substitution of $u^1(x, \rho)$ as given by Equation (6.2) it can be readily verified that $\dot{V}_1(x, \rho) = -1$. Since neither $V_1(x, \rho)$ nor $\dot{V}_1(x, \rho)$ are functions of r , it is obvious that the transition time from x^0 to the line $V_1(x, \rho) = 0$ is independent of r .

From the argument above, one may conclude that for any control u only that portion of u which is parallel to $u^1(x, \rho)$ is important in determining the transition time from an initial point to the line $V_1(x, \rho) = 0$. One may draw a similar conclusion for u^2 and V_2 .

For some initial state, x^0 , let the magnitude of the optimal control vectors, $u^1(x^0, \rho_1)$ and $u^2(x^0, \rho_2)$, be chosen such that $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = T$. Therefore the control vector $u^1(x^0, \rho_1)$ transfers the system from x^0 to the line $V_1 = 0$ in T seconds. Now consider a vector u such that the

portion of u which is parallel to u^1 is equal to $u^1(x^0, \rho_1)$. See Figure 6.3 for a graphical representation of this situation. The magnitude, ρ_1 , of the portion of u which is parallel to u^1 may be obtained from the scalar product of u and a unit vector parallel to u^1 . Hence

$$\rho_1 = u \cdot u^1(x^0, 1) \quad (6.7)$$

Since only the portion of u which is parallel to u^1 has any effect on the time necessary to transfer the system from x^0 to $V_1 = 0$, u must transfer the system from x^0 to both $V_1 = 0$ and $V_2 = 0$ in the same time. But $V_1 = V_2 = 0$ can only occur at the origin; hence u must transfer the system from x^0 to the origin in $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = T$ seconds.

Since setting $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$ specifies only the relative magnitude of ρ_1 in terms of ρ_2 , there are an infinite number of vectors which satisfy this condition. However, only one of these vectors has unit length. This is then an admissible control which transfers the system from x^0 to the origin in a finite time, $V_1(x^0, \rho_1)$. This is, in general, not the minimum time, but it is an acceptable compromise between system complexity and speed of response, as is shown later.

Several significant aspects of this sub-optimal control method should be noted. First, once the control is obtained, it is constant for the entire transition time.

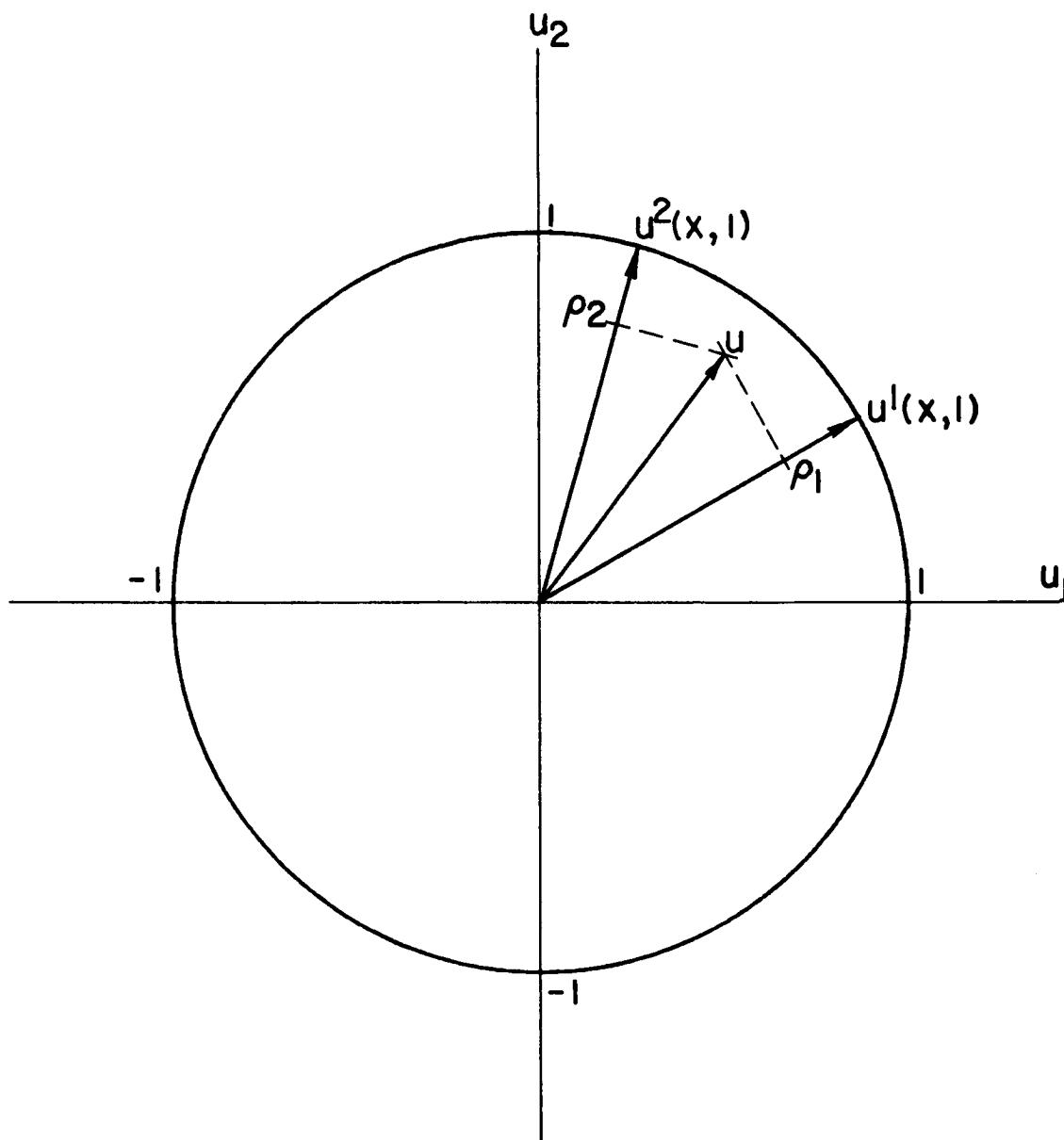


Figure 6.3 Representation of a Vector in Terms of the Optimal Control Vectors

For small disturbances, the control varies only slightly, which is helpful in mechanizing the controller. The control vector does not require rapid variations after its initial setting and hence only a minimum of recalculation of the control vector is necessary during the transition time. This should simplify the instrumentation of the controller.

Second, by the use of this method the transition time from any point to the origin can be easily obtained. This may be done by first setting the norm of the control vector equal to unity and then solving $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = T$ for T , which is the desired transition time. Isochrones can be found by choosing a value of T and finding the set of all points x such that $V_1(x, \rho_1) = V_2(x, \rho_2) = T$ and $\|u\| = 1$. An interesting aspect of such isochrones is that for a given T , they are quadratic in terms of x_1 and x_2 .

This method also makes it possible to obtain easily and directly the actual trajectory of the system from x^0 to the origin. This can be done in the following manner. After finding the transition time, T_0 , as described above, choose any time $T < T_0$; then solve for the point x such that $V_1(x, \rho_1) = V_2(x, \rho_2) = T$ with ρ_1 and ρ_2 as given above. This is the state of the system at T seconds before reaching the origin, or $T_0 - T$ seconds after leaving the initial state. This allows one to obtain the position of the system at any time during the transition to the origin with no knowledge of any previous state, thus eliminating any accumulation of

error. The work involved is strictly of an algebraic nature; it is not necessary to solve any differential equation. Both of these last two aspects of the method aid one in evaluating whether the performance of the sub-optimal system is satisfactory.

One further aspect of this method should be mentioned because of its importance relative to the implementation of the method. The simultaneous solution of the equations $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$ and $\|u\| = 1$ is an algebraic manipulation, although it is not trivial. This should be contrasted to many of the presently advocated methods for which it is necessary to solve simultaneously the usual nonlinear differential equations of the two point boundary value nature. The computational advantage is obvious from a hardware standpoint. Since these computations are algebraic, it is possible to carry them out continuously on an analog computer to create continuous control.

Before considering a numerical example to illustrate the method, it is perhaps of value to outline the complete method for reference.

- 1) Obtain the eigenvalues and eigenvectors of the matrix \underline{A}' .
- 2) Obtain the two Liapunov functions as given by Equation (6.1), $V_1(x, \rho)$ and $V_2(x, \rho)$, and their corresponding optimal controls, $u^1(x, \rho)$ and $u^2(x, \rho)$.

- 3) For a given point, x^0 , solve the relations
 $\rho_1 = u^1 u^1(x^0, 1)$ to obtain u_1 and u_2 in terms of
 ρ_1 and ρ_2 .
- 4) Solve $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$ and $\|u\| = 1$
simultaneously to obtain ρ_1 and ρ_2 .
- 5) By the use of the relations obtained in Step
Three, find u , the desired sub-optimal control.

A method of mechanizing the last three steps of this procedure by the use of a digital or analog computer to create a closed-loop system is shown schematically in Figure 6.4.

Two points should be emphasized again. First, once the control is determined, it remains relatively constant. Second, the operations required of the computer are strictly algebraic. It should also be noted that although the procedure is given in a step-by-step fashion, the control can be computed continuously by the use of an analog computer.

Example 6.1 In order to illustrate the method of sub-optimal control developed above, consider the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (6.9)$$

It is desired to transfer the system from the point $x^0 = (2, 1)$ to the origin with $\|u\| \leq 1$.

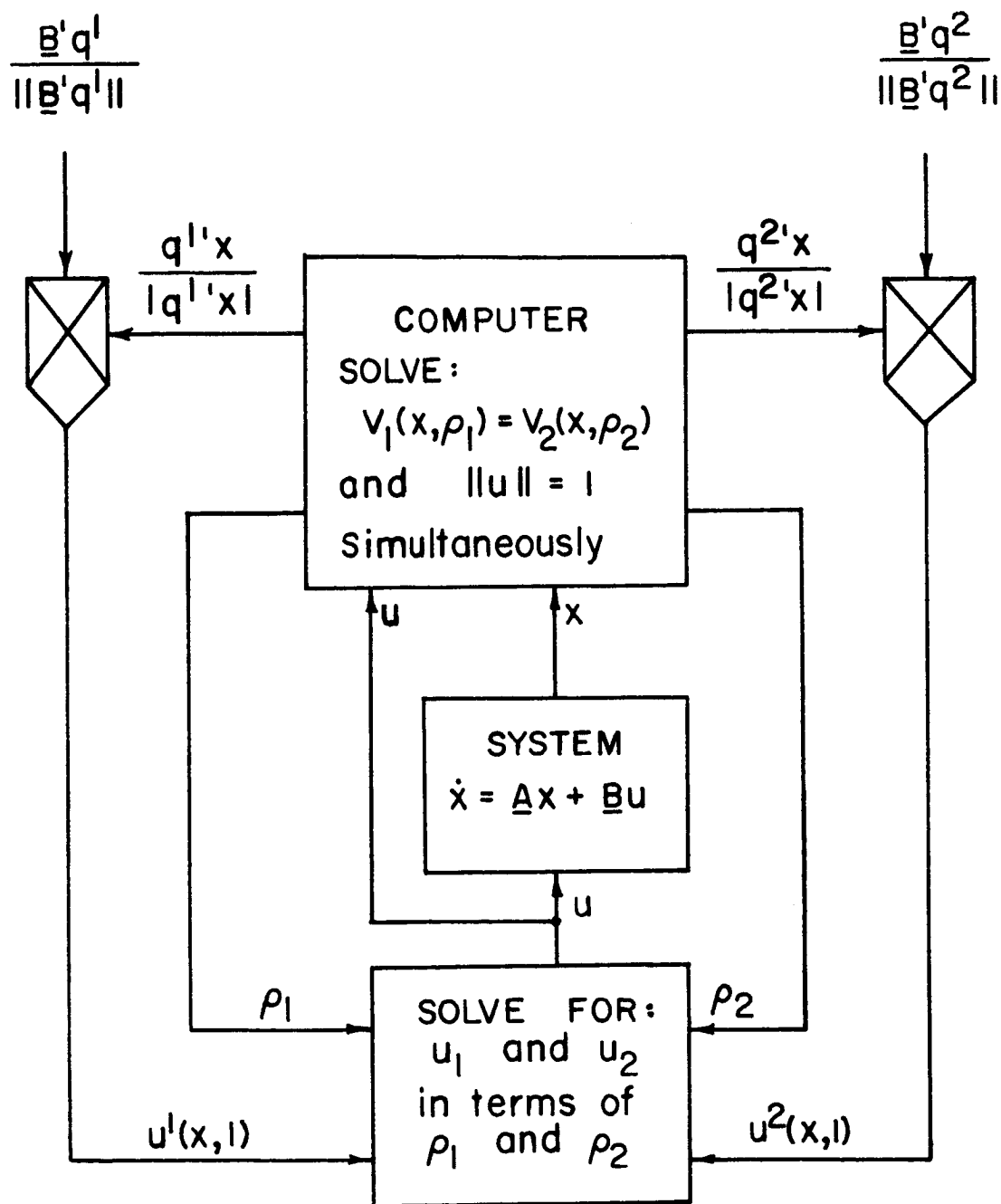


Figure 6.4 Closed-Loop, Sub-Optimal Control System

By standard methods the eigenvalues are found to be -1 and -2 with the corresponding eigenvectors of \underline{A}' being (2,1) and (1,1). The two Liapunov functions as given by Equation (6.1) are

$$V_1(x, \rho) = \ln \left(\frac{|2x_1 + x_2|}{2\rho} + 1 \right) \quad (6.10)$$

$$V_2(x, \rho) = \frac{1}{2} \ln \left(\frac{4|x_1 + x_2|}{\sqrt{5}\rho} + 1 \right) \quad (6.11)$$

The corresponding optimal controls as given by Equation (6.2) are

$$\begin{bmatrix} u_1^1(x, \rho) \\ u_2^1(x, \rho) \end{bmatrix} = -\rho \left(\frac{2x_1 + x_2}{|2x_1 + x_2|} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (6.12)$$

$$\begin{bmatrix} u_1^2(x, \rho) \\ u_2^2(x, \rho) \end{bmatrix} = -\rho \left(\frac{x_1 + x_2}{|x_1 + x_2|} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad (6.13)$$

This completes Steps One and Two in the procedure outlined above.

Now for $x^0 = (2, 1)$, $u^1(x^0, 1)$ and $u^2(x^0, 1)$ become

$$u^1(x^0, 1) = (-1, 0) \quad (6.14)$$

$$u^2(x^0, 1) = (-2/\sqrt{5}, -1/\sqrt{5}) \quad (6.15)$$

Using the relation $\rho_1 = u'u^1(x^0, 1)$, one obtains

$$\rho_1 = -u_1$$

$$\rho_2 = -2u_1/\sqrt{5} - u_2/\sqrt{5}$$

Then solving for u_1 and u_2 in terms of ρ_1 and ρ_2 yields

$$u_1 = -\rho_1$$

$$u_2 = +2\rho_1 - \sqrt{5}\rho_2$$

Now setting $\|u\| = 1$, one obtains

$$(-\rho_1)^2 + (-2\rho_1 + \sqrt{5}\rho_2)^2 = 1 \quad (6.16)$$

Setting $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$, one obtains

$$\ln \left(\frac{5}{2\rho_1} + 1 \right) = \frac{1}{2} \ln \left(\frac{12}{\sqrt{5}\rho_2} + 1 \right) \quad (6.17)$$

By solving Equations (6.16) and (6.17) simultaneously, it is possible to obtain ρ_1 and ρ_2 . If this is done the solution obtained is

$$\rho_1 = 0.645$$

$$\rho_2 = 0.236$$

Therefore a control vector $u = (-0.645, +0.763)$ transfers the system from the point $x^0 = (2, 1)$ to the origin in $V_1(x^0, \rho_1) = 1.58$ seconds.

6.3 Generalization

In the previous section a sub-optimal method was developed for second-order systems. A generalization of

this method to n -th order systems is presented in this section. Since all of the proofs and arguments carry over directly to the n -th order case, only the conclusions are presented here.

In the n -th order case, there are n real and distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and hence n linearly independent eigenvectors q^1, q^2, \dots, q^n . Associated with each eigenvector is a Liapunov function given by Equation (6.1), $V_1(x, \rho), V_2(x, \rho), \dots, V_n(x, \rho)$ and their corresponding optimal controls $u^1(x, \rho), u^2(x, \rho), \dots, u^n(x, \rho)$. As before, for some point in the state space, x^0 , the control given by $u^1(x, \rho)$ transfers the system from x^0 to some point, x^1 , on the hyperplane $V_1(x, \rho) = 0$ in minimum time. Again the control $u^1(x, \rho)$ is constant during the entire transition time.

By an argument identical to that presented in the previous section, it can be shown that for any control u only that portion of u which is parallel to $u^1(x, \rho)$ affects the transition time from any initial point to the hyperplane $V_1(x, \rho) = 0$.

If for some initial state x^0 a control vector u is chosen such that $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = \dots = V_n(x^0, \rho_n)$ where $\rho_1 = u^1(x, 1)$, then this control must transfer the system to the origin in $V_1(x^0, \rho_1)$ seconds. Since there are only $n-1$ equations in the n unknowns, $\rho_1, \rho_2, \dots, \rho_n$, there is an infinite set of control vectors which satisfy these equations. From this set, there is only one vector whose

norm is equal to unity. This is an admissible control which transfers the system from any point x^0 to the origin in a finite, although usually not minimum, time.

Then the procedure for obtaining a sub-optimal control can be stated in the following steps.

- 1) Obtain the eigenvalues and eigenvectors of the matrix A' .
- 2) Obtain the Liapunov functions as given by Equation (6.1), $V_1(x, \rho)$, $V_2(x, \rho)$, \dots , $V_n(x, \rho)$ and their corresponding optimal controls, $u^1(x, \rho)$, $u^2(x, \rho)$, \dots , $u^n(x, \rho)$.
- 3) For a given point, x^0 , solve the relations $\rho_1 = u^1(x^0, 1)$ to obtain u_1 in terms of the ρ_1 .
- 4) Solve $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = \dots = V_n(x^0, \rho_n)$ and $\|u\| = 1$ simultaneously to obtain $\rho_1, \rho_2, \dots, \rho_n$.
- 5) By the use of the relations obtained in Step Three, find u , the desired sub-optimal control.

As before the last three steps in this procedure can be mechanized by the use of a digital or analog computer in order to create a closed-loop system.

All of the features of this method which were pointed out for second-order systems carry over directly for n -th order systems.

Example 6.2 As an illustration of the above procedure, consider the third-order system shown in Figure 6.5.

The equations of motion may be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

It is desired to find a sub-optimal control which transfers the system from the point $x^0 = (1, 2, 3)$ to the origin. The norm of the control vector is constrained to be equal to or less than unity.

The eigenvalues are -1, -2, and -3 with the corresponding eigenvectors of A' being $(0, 0, 1)$, $(0, 1, -1)$ and $(2, -2, 1)$. The Liapunov functions are found from Equation (6.1) to be

$$V_1(x, \rho) = \ln \left(\frac{|x_3|}{\rho} + 1 \right)$$

$$V_2(x, \rho) = \frac{1}{2} \ln \left(\frac{\sqrt{2}|x_2 - x_3|}{\rho} + 1 \right)$$

$$V_3(x, \rho) = \frac{1}{3} \ln \left(\frac{|2x_1 - 2x_2 + x_3|}{\rho} + 1 \right)$$

The corresponding optimal controls are then

$$\begin{bmatrix} u_1^1(x, 1) \\ u_2^1(x, 1) \\ u_3^1(x, 1) \end{bmatrix} = - \begin{bmatrix} 0 \\ \frac{x_3}{|x_3|} \\ 1 \end{bmatrix}$$

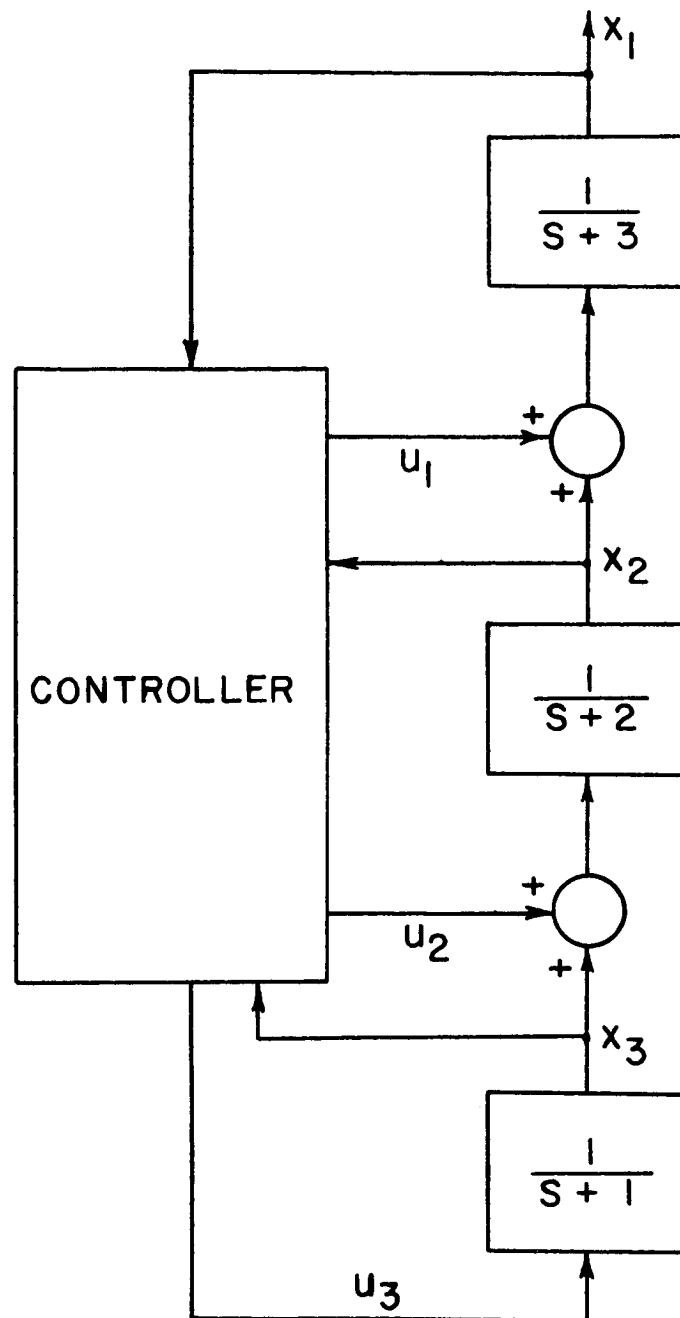


Figure 6.5 Third Order Example for Sub-Optimal Control with Non-Singular Control Matrix

$$\begin{bmatrix} u_1^2(x,1) \\ u_2^2(x,1) \\ u_3^2(x,1) \end{bmatrix} = - \frac{(x_2 - x_3)}{|x_2 - x_3|} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} u_1^3(x,1) \\ u_2^3(x,1) \\ u_3^3(x,1) \end{bmatrix} = - \frac{(2x_1 - 2x_2 + x_3)}{|2x_1 - 2x_2 + x_3|} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

By the use of the initial state $x^0 = (1,2,3)$ and the relations $\rho_1 = u'u^1(x^0,1)$, it is possible to solve for the components of the control vector in terms of the ρ_1 's.

$$u_1 = -\rho_1/2 + \sqrt{2} \rho_2 - 3\rho_3/2$$

$$u_2 = -\rho_1 + \sqrt{2} \rho_2$$

$$u_3 = -\rho_1$$

By setting $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = V_3(x^0, \rho_3)$, one obtains the following two equations

$$\ln\left(\frac{3}{\rho_1} + 1\right) = \frac{1}{2}\ln\left(\frac{\sqrt{2}}{\rho_2} + 1\right) \quad (6.18)$$

$$\ln\left(\frac{3}{\rho_1} + 1\right) = \frac{1}{3}\ln\left(\frac{1}{\rho_3} + 1\right) \quad (6.19)$$

In order to obtain the desired control vector, it is necessary to solve Equations (6.18) and (6.19) simultaneously with the condition $\|u\| = 1$. The answers that one obtains are

$$\rho_1 = 0.714$$

$$\rho_2 = 0.0543$$

$$\rho_3 = 0.00716$$

The desired sub-optimal control is then

$$u(x^0) = (-0.291, -0.637, -0.714)$$

This control transfers the system from $x^0 = (1, 2, 3)$ to the origin in 1.65 seconds.

6.4 Discussion of the Method

In the previous sections, a method of obtaining sub-optimal control of systems in which the control matrix is non-singular was developed. Every sub-optimal control method should satisfy two requirements. First, the method should make it easy to design and implement the sub-optimal control system. Second, the performance of the sub-optimal control system should be acceptably close to the true optimum.

This method has several aspects which assist in the design and implementation of the sub-optimal system. These points have been discussed in Section 6.2, but they are repeated here for reference. In the absence of a disturbance the control vector, once obtained, remains constant until the system reaches the origin. The transition time from any point to the origin as well as the trajectory to the origin can be obtained readily. The isochrones are easy to find. In designing a closed-loop control system using

this method, it is necessary for the controller-computer to solve only algebraic equations, thus allowing continuous control.

Until now the quality of the performance of the sub-optimal system has been ignored. It is shown in this section that the quality is acceptable. Because of the difficulty involved, it is not possible to obtain the true optimal solution, and hence it is necessary to use the approach discussed in Section 5.4. In particular, it is shown that the sub-optimal isochrone, S_1 , is tangent to the optimal isochrone, S_0 , at several points.

As was pointed out in Section 5.4, there must be two points on the $V_1(x,1) = T_0$ surface from which the origin is reached in T_0 seconds as a special case of reaching the surface $V_1 = 0$ using the control $u = u^1(x,1)$. Since the system reaches the origin in T_0 seconds, it must also reach all of the $V_1(x,1) = 0$ surfaces in the same time. Hence $V_1(x^0,1) = V_2(x^0,\rho_2) = \dots = V_n(x^0,\rho_n)$, where $\rho_1 = u^1(x^0,1) \cdot u^1(x^0,1)$, is satisfied at this point. The control $u^1(x^0,1)$ therefore satisfies all of the conditions of the sub-optimal control, and hence it is the sub-optimal control for these points. Therefore these points must be on the sub-optimal isochrone. But it is shown in Section 5.4 that these points are also on the S_0 surface. Hence the sub-optimal and optimal isochrones must be tangent at these points.

In a similar fashion one could argue that there are two points on each $V_1(x,1) = T_0$ surface which are on both the optimal and sub-optimal isochrones. Hence there must be $2n$ points at which these surfaces are tangent. Since both of the surfaces are smooth, it is logical to assume that they are close in some region about each of these points.

One could get a direct measure of the quality of the sub-optimal control by determining the optimal isochrones for particular problems such as the ones in Examples 6.1 and 6.2. However, the advisability of this is highly questionable. First, as was pointed out in Chapter 2, the computational labor involved in obtaining the optimum solution for even one point is horrendous for all but trivial problems. To find a complete set of such points is almost unthinkable. Second, if one were to carry out such computations, the most that one could conclude would be that the sub-optimal method was good or bad for that particular example.

It appears reasonable from the above points to conclude that this sub-optimal control method represents an acceptable compromise between system complexity and speed of response.

Although the method presented in this chapter is significant and important in its own context, its major significance is in providing an underlying framework for the next chapter. In the case of non-singular \underline{B} matrix, several other sub-optimal methods have been suggested. None of

these methods, however, has, as of yet, produced a sub-optimal control better than that presented here. The number of practical systems for which \underline{B} is non-singular is limited, and hence the material in the next chapter is of greater practical significance.

CHAPTER 7

SUB-OPTIMAL CONTROL WITH SINGULAR CONTROL MATRIX

7.1 Introduction

In the previous chapter a method for using the eigenvector scalar product solutions for the sub-optimal control of systems with non-singular control matrices was developed. This restriction on the control matrix is removed in this chapter. The result of this change is that it is no longer possible to develop a single method which handles all problems. Hence the approach taken is to develop two methods, each of which has special restrictions. By carefully delineating the range of applicability, the advantages and the disadvantages of each method, it is possible to choose the method or the combination of methods which best applies to a given problem.

In addition to the control matrix being singular, it is assumed that the dimension of the control space, r , is equal to the rank of the matrix \underline{B} which is less than the dimension of the state space, n . If this is not true, then one may always reduce the number of control variables in the following manner.

Since the control matrix, \underline{B} , is singular and n by n , there must be at least one column of \underline{B} which can be formed by a linear combination of the other columns. For

convenience let the last column be formed by a linear combination of the previous columns. Then

$$b_{1n} = \sum_{j=1}^{n-1} c_j b_{1j} \quad (7.1)$$

Therefore Equation (2.2) becomes

$$\begin{aligned} \dot{x}_1 &= \sum_{j=1}^n a_{1j} x_j + \sum_{j=1}^{n-1} b_{1j} u_j + \sum_{j=1}^{n-1} c_j b_{1j} u_n \\ &= \sum_{j=1}^n a_{1j} x_j + \sum_{j=1}^{n-1} b_{1j} (u_j + c_j u_n) \end{aligned} \quad (7.2)$$

Since u_n only appears in terms of $(u_j + c_j u_n)$, let

$$w_j = u_j + c_j u_n \quad j = 1, 2, \dots, n-1 \quad (7.3)$$

In matrix form Equation (7.3) may be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 1 & 0 & 0 & \cdots & c_2 \\ \vdots & & \cdots & 1 & 0 & \vdots \\ 0 & & \cdots & 0 & 1 & c_{n-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{bmatrix} \quad (7.4)$$

Since the coefficient matrix on the left is of rank $n-1$ and has only $n-1$ rows, the coefficient matrix and the augmented coefficient matrix must be of the same rank. Therefore it is possible to solve for the u 's in terms of the w 's. In fact, since the rank is only $n-1$, there are an infinite

number of solutions. Therefore, there exists a matrix \underline{D} such that

$$u = \underline{D}w \quad (7.5)$$

Then the requirement on the norm of u becomes

$$\|u\|^2 = w' \underline{D}' \underline{D} w \leq 1 \quad (7.6)$$

The matrix $\underline{D}' \underline{D}$ is an $n-1$ by $n-1$ positive definite matrix; therefore there is an $n-1$ by $n-1$ non-singular matrix \underline{E} such that $\underline{E}' \underline{E} = \underline{D}' \underline{D}$. Now let a new set of $n-1$ control variables be defined as

$$v = \underline{E}w$$

Then $w' \underline{D}' \underline{D} w = w' \underline{E}' \underline{E} w = v' v = \|v\|^2 \leq 1$. Hence a set of $n-1$ control variables has been generated with the requirement that its norm be less than or equal to unity. Equation (2.2) has now become

$$\dot{x} = \underline{A}x + \underline{B} \underline{D} \underline{E}^{-1} v \quad (7.7)$$

This process can be repeated until the number of control variables is equal to the rank of \underline{B} .

It is noted in Chapter 5 that the eigenvector scalar product solutions do not require a non-singular control matrix; these solutions are therefore used in creating the sub-optimal methods of this chapter. The sub-optimal control method for systems with non-singular control matrices is based on the concept of picking a control vector, u , such

that the n equations $V_1(x, \rho_1) = V_2(x, \rho_2) = \dots = V_n(x, \rho_n)$ and $\|u\| = 1$ are simultaneously satisfied where $\rho_1 = u^*u^1(x, 1)$. It is possible to obtain a solution to these n equations because there are n control variables.

It is not possible to apply such an approach directly for the systems being considered in this chapter, since the number of control variables is less than n . Hence if the control space is r -dimensional, it is possible to solve only r of these equations simultaneously. Thus the method of the previous chapter must be modified.

One approach is to simply disregard $n-r$ of the equations. If, for example, only the first r Liapunov functions are equated while maintaining the requirement on the norm of u , then it is possible to satisfy this set of r equations. The consequence is that the control chosen drives the system to the state $V_1 = V_2 = \dots = V_r = 0$, while the remaining $n-r$ Liapunov functions are, in general, non-zero. This concept forms the basis of the method presented in Section 7.2.

Another approach offers perhaps the best solution, although it is the most difficult to apply. It is shown later that the set of all states such that the n equations $V_1 = V_2 = \dots = V_n$ and $\|u\| = 1$ can be simultaneously satisfied by an r -dimensional control, is an r -dimensional subset, R , of the n -dimensional state space. It is obvious that the origin must be included in this subset. Then from any point in R it is possible to transfer the system to the origin in

finite time with an r -dimensional constant control vector. However, since R is only r -dimensional, it is necessary to first transfer the system to this subset. In general it is necessary to make changes in the control vector in order to get into R . Once the system is in R , a constant control vector, u , can be determined, which transfers the system directly to the origin.

In the following sections each of the methods is discussed in detail including examples. As a final result, the methods are combined to illustrate the flexibility of the approach.

7.2 The Bang-Coast Method

In Chapter 6 a sub-optimal control method was developed based on the concept of finding a control vector with unit magnitude such that the time needed to reach each of the n surfaces of $V_1 = 0$ was equal. This technique required the simultaneous solution of the n algebraic equations $V_1(x, \rho_1) = V_2(x, \rho_2) = \dots = V_n(x, \rho_n)$ and $\|u\| = 1$. These n equations could be solved simultaneously because the control space was n -dimensional and hence there were n control variables.

For the work in this chapter, the control space is r -dimensional, where $r < n$, and therefore the direct solution of these n equations is not possible for an arbitrary system state. Hence some modification of the method of Chapter 6

is necessary. The approach taken in this section is based on the concept of equating only r of the eigenvector scalar product solutions to create $r-1$ equations. Combining these $r-1$ equations with the requirement on the control norm, one obtains r equations. Because there are r control variables, these r equations can be satisfied simultaneously. There are two questions which are raised by such an approach. First, what are the consequences of equating only r of the eigenvector scalar product solutions rather than all n ? Second, which eigenvector scalar product solutions should be chosen? Before answering these questions, it is convenient to introduce another class of scalar functions.

Consider for the moment the case where $u = 0$ and Equation (2.2) becomes

$$\dot{\mathbf{x}} = \mathbf{Ax}$$

Now define a set of scalar functions, $Z_i(\mathbf{x})$, by

$$Z_i(\mathbf{x}) = \mathbf{q}^i \cdot \mathbf{x} \quad i = 1, 2, \dots, n$$

Then by application of Theorem 5.3, the total time derivative of $Z_i(\mathbf{x})$ is given by

$$\dot{Z}_i(\mathbf{x}) = \lambda_i Z_i(\mathbf{x}) \quad (7.8)$$

Given the state of the system at $t = T$, the value of each $Z_i(\mathbf{x})$ function at $t = T$ is uniquely given by $Z_i(\mathbf{x}(T)) = \mathbf{q}^i \cdot \mathbf{x}(T)$. The value of each $Z_i(\mathbf{x})$ for any time $t \geq T$ can be

found by solving the simple first-order linear differential Equation (7.8) with the above initial conditions.

$$Z_1(x(t)) = q^1 x(t) = Z_1(x(T)) e^{\lambda_1(t-T)} \quad (7.9)$$

Thus given the value of any $Z_1(x)$ at $t = T$, its value at any time $t \geq T$ is uniquely determined by the use of Equation (7.9).

Now consider again the idea of equating only r of the eigenvector scalar product solutions. Since the ordering of the functions is arbitrary, for convenience, the first r functions are equated.

If for an initial state, x^0 , the r equations $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = \dots = V_r(x^0, \rho_r)$ and $\|u\| = 1$ where $\rho_1 = u^1 u^1(x^0, 1)$ are solved simultaneously, then a control vector u is obtained which transfers the system from x^0 to a state x^1 such that $V_1(x^1) = V_2(x^1) = \dots = V_r(x^1) = 0$ in time $T = V_1(x^0, \rho_1)$. In general the values of the remaining $n-r$ eigenvector scalar product solutions are non-zero at x^1 . Hence the system has not been transferred to the origin and further action is necessary.

It can be readily observed from Equation (6.1) that if $V_1(x, \rho_1)$ is zero, then $q^1 x$ is zero, and hence $Z_1(x)$ is also zero. Conversely if $Z_1(x)$ is zero, then $V_1(x, \rho_1)$ is zero. Hence at the state x^1 , $Z_1 = Z_2 = \dots = Z_r = 0$ while the values of the remaining Z 's are non-zero.

If, after reaching the state x^1 where the values of the first r eigenvector scalar product solutions are zero, the control vector u is set to zero, then the value of each $Z_i(x)$ for $t \geq T$ is given by Equation (7.9). Obviously for $t \geq T$, $Z_1 = Z_2 = \dots = Z_r = 0$, and hence the system remains on the $V_1 = V_2 = \dots = V_r = 0$ surface. The value of each of the other Z_i , $i = r+1, r+2, \dots, n$ is exponentially approaching zero with a time-constant equal to $1/|\lambda_i|$. This can only be true if the system is exponentially approaching the origin. However, the rate at which the system approaches the origin is dependent on the longest time-constant, i.e. the smallest value contained in the set $|\lambda_{r+1}|, |\lambda_{r+2}|, \dots, |\lambda_n|$.

Which eigenvector scalar product solutions to equate is now obvious. First let the eigenvector scalar products be ordered such that $|\lambda_i| < |\lambda_j|$ if $i < j$. Then equating the first r eigenvector scalar product solutions to obtain the control u results in the system approaching the origin with a time-constant equal to $1/|\lambda_{r+1}|$. No other choice of eigenvector scalar product solutions can make the system approach the origin faster.

The complete procedure then is the following. First obtain the r eigenvector scalar product solutions associated with the r smallest eigenvalues. Then for an arbitrary initial state, x^0 , solve simultaneously the r equations $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = \dots = V_r(x^0, \rho_r)$ and $\|u\| = 1$

simultaneously as outlined in Chapter 6. Use the control u to transfer the system from x^0 to x^1 where $V_1(x^1, \rho_1) = \dots = V_r(x^1, \rho_r) = 0$. At x^1 the control is turned off and the system is allowed to coast uncontrolled toward the origin with a time-constant equal to $1/|\lambda_{r+1}|$. Again this procedure can be mechanized in a manner similar to that presented in Chapter 6. The only difference is that when the state of the system reaches the surface $V_1 = V_2 = \dots = V_r = 0$, the control is turned off.

The method of sub-optimal control presented in this section is most advantageously applied in cases where the magnitudes of r or less of the eigenvalues are small relative to the remaining eigenvalues. If this is the case, during the controlled portion of the response, the small eigenvalues can be effectively eliminated from the system by driving the system to the surface $V_1 = V_2 = \dots = V_r = 0$. The system then approaches the origin with only the relatively fast time-constants present.

As an example, consider a third-order system with a two-dimensional control space and eigenvalues of -1, -2, and -10. By selecting a control vector such that for some initial state, x^0 , $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$ and $\|u\| = 1$, then the system is transferred to a state where $V_1 = V_2 = 0$. The system then approaches the origin with only the time-constant 1/10 present, thus yielding relatively rapid response.

One further aspect of this method should also be mentioned. Although the method was initially developed to yield approximately time-optimal performance, the method appears to present a good compromise between time- and fuel-optimal performance. The amount of fuel expended is roughly proportional to $\int_0^{\infty} \|u\| dt$. Hence during the controlled portion of the response, the fuel expenditure is high while after the turn-off of the control, the fuel expenditure is zero. This is characteristic of time- fuel-optimal performance. Hence the method presents a distinct advantage in cases where the total fuel expenditure is of importance.

7.3 The Unbounded Control Method

In the previous section the inability to solve n equations with only r control variables was bypassed by equating only r of the eigenvector scalar product solutions. It seems reasonable to conclude that if the restriction on the norm of the control vector were removed, $r+1$ of the eigenvector scalar product solutions could be equated. Although a control vector with a norm greater than one violates one of the basic assumptions of this work, the concept is still of theoretical, if not practical, importance. However, as is shown by the example below, removing the restriction on the control norm does not allow $r+1$ eigenvector scalar product solutions to be equated at an arbitrary system state.

Example 7.1 Consider again the second-order system with scalar control discussed in Example 5.1.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (7.10)$$

The eigenvector scalar product solutions of the Hamilton-Jacobi equation as found in Chapter 5 are

$$V_1(x, \rho_1) = \ln \left(\frac{|2x_1 + x_2|}{\rho_1} + 1 \right) \quad (7.11)$$

$$V_2(x, \rho_2) = \frac{1}{2} \ln \left(\frac{2|x_1 + x_2|}{\rho_2} + 1 \right) \quad (7.12)$$

with the associated optimal controls

$$u^1(x, \rho_1) = -\rho_1 \frac{(2x_1 + x_2)}{|2x_1 + x_2|} \quad (7.13)$$

$$u^2(x, \rho_2) = -\rho_2 \frac{(x_1 + x_2)}{|x_1 + x_2|} \quad (7.14)$$

Let the initial state of the system be $x^0 = (2, -3)$. Then $u^1(x^0, 1) = -1$ while $u^2(x^0, 1) = +1$. Hence no control vector, independent of its magnitude, could cause both ρ_1 and ρ_2 to be positive. Therefore it is impossible to satisfy the equation $V_1(x^0, \rho_1) = V_2(x^0, \rho_2)$ with any control vector. It is obvious that any state in the region where $u^1(x, 1) \neq u^2(x, 1)$ exhibits this same difficulty.

However, it is not possible to satisfy this equation at every state in the region where $u^1(x, 1) = u^2(x, 1)$. In order to see this fact, set $V_1(x, \rho_1)$ equal to $V_2(x, \rho_2)$ to obtain:

$$\ln \left(\frac{|2x_1 + x_2|}{\rho_1} + 1 \right) = \frac{1}{2} \ln \left(\frac{2|x_1 + x_2|}{\rho_2} + 1 \right) \quad (7.15)$$

Since the control is scalar, ρ_1 must be equal to ρ_2 ; therefore Equation (7.15) becomes

$$\ln \left(\frac{|2x_1 + x_2|}{\rho_1} + 1 \right) = \frac{1}{2} \ln \left(\frac{2|x_1 + x_2|}{\rho_1} + 1 \right) \quad (7.16)$$

Solving for ρ_1 , one obtains

$$\rho_1 = \frac{|2x_1 + x_2|^2}{2(|x_1 + x_2| - |2x_1 + x_2|)} \quad (7.17)$$

In order for ρ_1 to be positive, it is necessary that $|2x_1 + x_2| < |x_1 + x_2|$. Thus the region where it is possible to solve the equation $V_1(x, \rho_1) = V_2(x, \rho_2)$ is quite restricted. Therefore, in spite of removing the restriction on the norm of the control vector, it is only possible to equate $r+1$ of the eigenvector scalar product solutions in a restricted region of the state space.

The method presented in this section has three fundamental problems. First, removing the restriction on the control norm is a violation of one of the basic assumptions of this work. Second, even with unbounded control, it is not possible to equate $r+1$ of the eigenvector scalar product solutions for an arbitrary initial point. Third, at best, this technique would allow the system to be driven to a state where $r+1$ rather than r of the eigenvector scalar product solutions were zero. However, in the next section, another method of accomplishing this same result is

presented which retains the restriction on the control norm. With these fundamental problems, it is felt that this approach offers no significant advantages and hence further investigation of it seems unwarranted.

7.4 The Switched Control Method

In the previous sections the problem of satisfying n equations with an r -dimensional control vector was solved by simply disregarding or weakening $n-r$ of the equations. In this section the portion of the state space in which all n equations can be satisfied by an r -dimensional control vector is obtained. It is shown below that this region is an unbounded r -dimensional subset, R , of the n -dimensional state space. Since the subset R is only r -dimensional, an arbitrary initial state is not in R , and hence it is necessary to first transfer the system to R . Then from any point in R , it is possible to determine a constant control vector, u , which transfers the system to the origin in finite time.

The problem of finding a control vector which transfers a system from any point in R to the origin is exactly the same as the problem treated in Chapter 6. However, the problem of transferring the system from any arbitrary initial state to R is new. A complete solution of this problem for an arbitrary difference between n and r has not been obtained. In fact, only the case in which the

difference between n and r is equal to one has been solved. But before investigating this problem, consider again the determination of the subset R .

Let the eigenvector scalar product solutions be ordered as in Section 7.2, i.e. such that $|\lambda_1| < |\lambda_j|$ if $1 < j$. Since there are only r control variables, in order to solve simultaneously n equations there must exist $n-r$ relations among the state variables. Therefore the subset R must be r -dimensional. Thus due to the fact that there are only r control variables, the subset from which the origin can be reached by the use of a constant control vector is only r -dimensional.

Example 7.2 As an illustration of the determination of a subset R , consider again the system discussed in Example 7.1. It is not necessary to use the generalization of the control vector norm, since u is scalar, and hence both ρ_1 and ρ_2 are identically one if the norm of u is one. The eigenvector scalar product solutions then become

$$V_1(x) = \ln (|2x_1 + x_2| + 1) \quad (7.18)$$

$$V_2(x) = \frac{1}{2} \ln (2|x_1 + x_2| + 1) \quad (7.19)$$

In order to find the subset R , it is necessary to equate V_1 and V_2 to obtain

$$\ln (|2x_1 + x_2| + 1) = \frac{1}{2} \ln (2|x_1 + x_2| + 1) \quad (7.20)$$

and then to solve for the system state for which this equation is satisfied. If this equation is solved directly, two one-dimensional subsets are obtained. However, only one of these subsets is the true R subset from which it is possible to reach the origin by the use of a constant control vector. For the states comprising the other subset, the two optimal control vectors, u^1 and u^2 , have opposite directions, and hence it is not possible to transfer the system from these states to the origin directly. Figure 7.1 shows a graphical representation of both the true and false R subsets.

The reader may recognize the R subset as the time-optimal switching curve as obtained by running time backwards. This fact, which can be easily verified, plays an important role in the development of a method for transferring the system from any initial state to the subset R . It should be noted that by the use of the above method a non-parametric representation of the switching curve is obtained.

It should be noted that in the application of the procedure developed below, it is not necessary to determine the subset R explicitly as presented above. Inherent in the method of transferring the system to the subset R is an ability to determine when the system is in R . The subset R was determined for the above problem, since it is helpful in discussing the procedure for transferring the system from an arbitrary initial state to R .

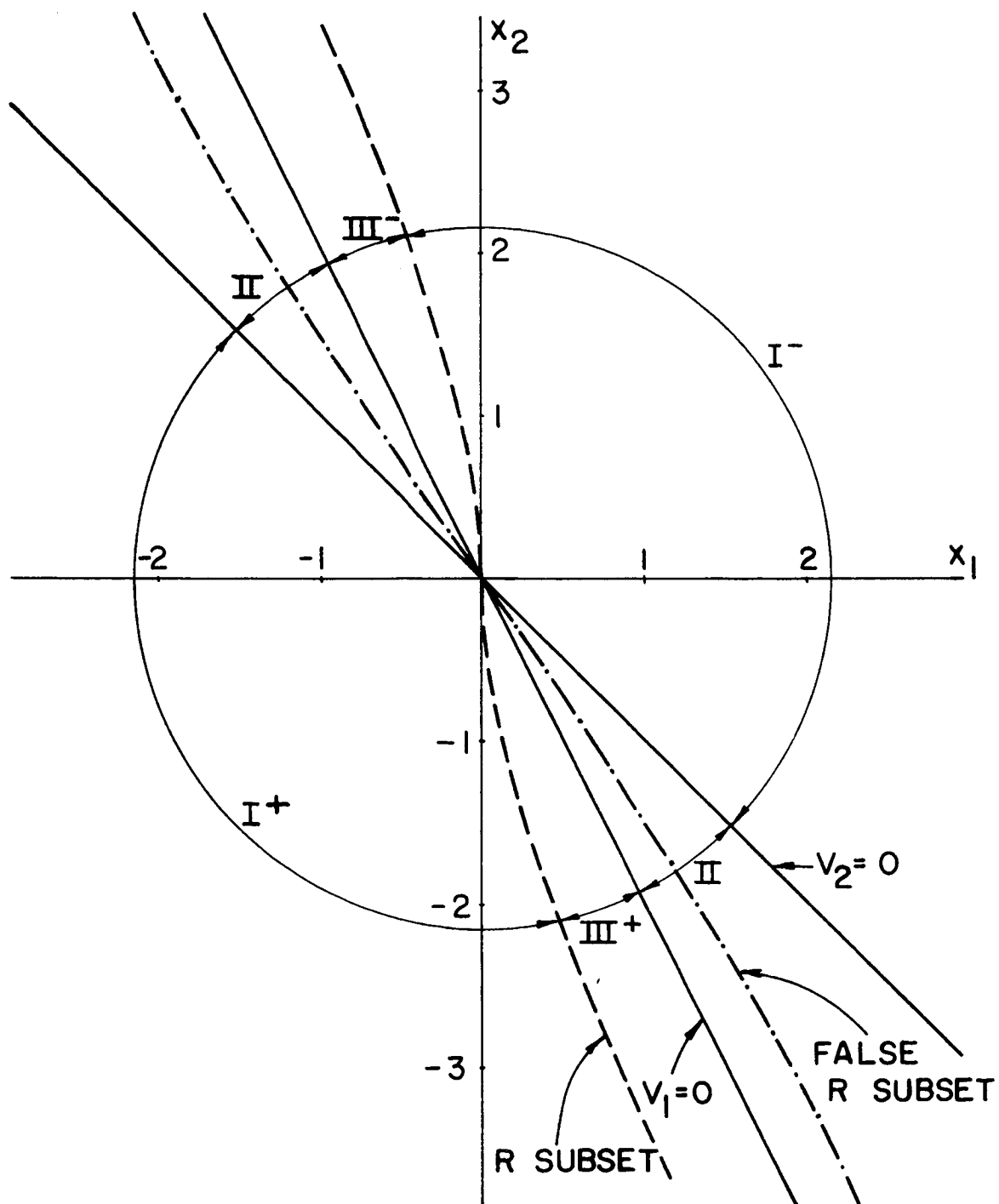


Figure 7.1 R Subset for a Second-Order System

Now consider the division of the state space into the following three regions as shown on Figure 7.1.

Region I: $u^1 = u^2$ and $V_1 \geq V_2$.

Region II: $u^1 \neq u^2$.

Region III: $u^1 = u^2$ and $V_1 \leq V_2$.

It should be noted that the lines of division between these regions are $V_1 = 0$ lines. Regions I and III are further subdivided into positive and negative regions, depending on whether u^1 is positive or negative. See Figure 7.1.

Since this problem is only second-order with scalar control, the true time-optimal solution may be obtained by standard techniques (Pontryagin 1962). If this is done for some initial state, x^0 , then the optimal trajectory obtained is shown in Figure 7.2. The corresponding optimal control, u^0 , is equal to -1 from x^0 to x^c and +1 from x^c to the origin. Again it should be noted that the point x^c at which the change from -1 to +1 occurs is in the subset R. Hence the optimum trajectory is, in fact, a path from any initial state to R and thence to the origin. Therefore by investigating more closely the optimal control from x^0 to x^c in terms of the eigenvector scalar product solutions, perhaps a procedure can be developed for transferring the system from an initial state to R.

It can be readily seen that the optimal control, u^0 , is equal to u^1 in Regions I and II. However, in Region III, u^0 is equal to $-u^1$. By the use of the description of these

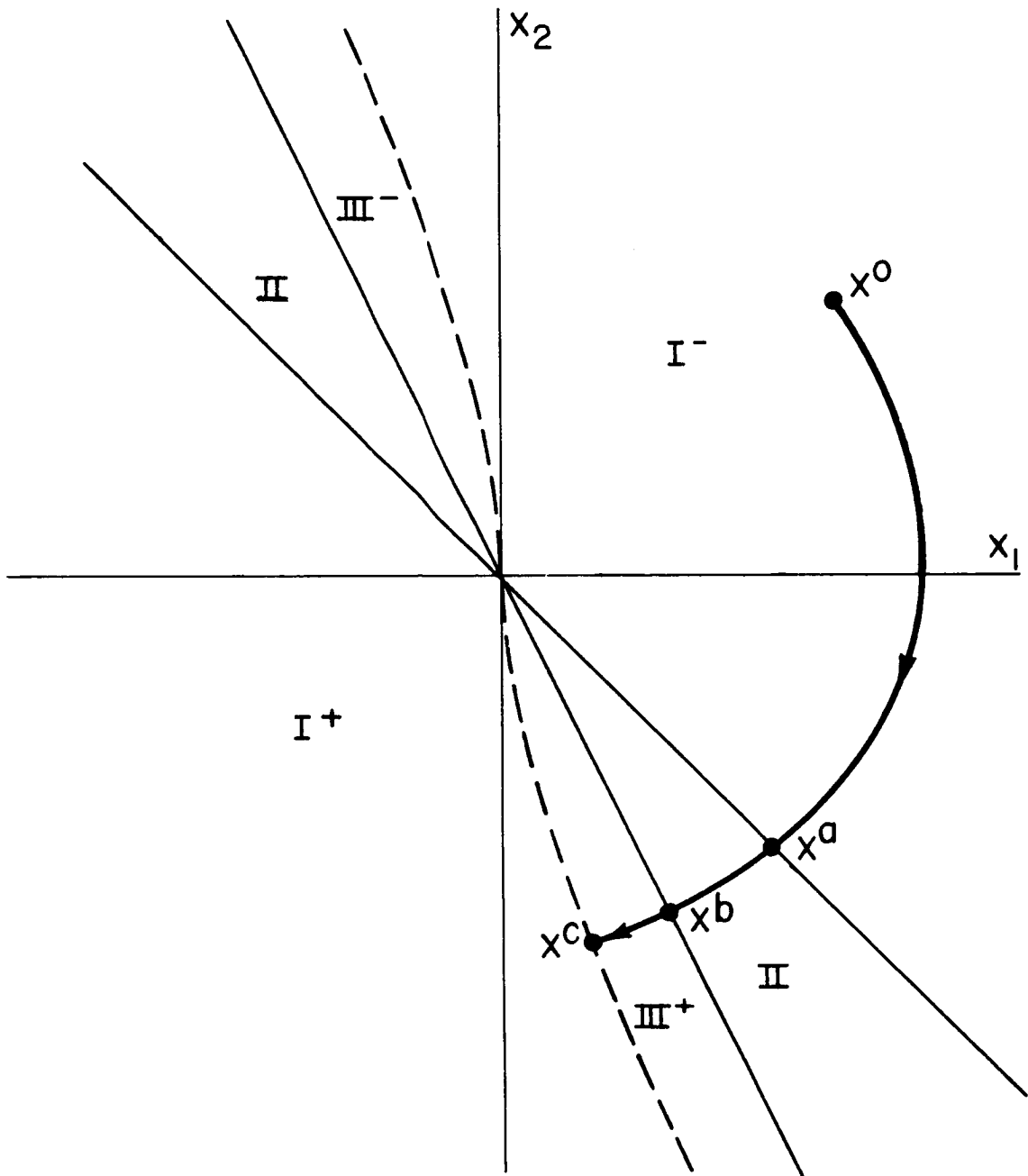


Figure 7.2 Second-Order, Time-Optimal Trajectory

regions, one can formulate the optimal control policy in terms of the eigenvector scalar product solutions as the following.

- 1) If $u^1 \neq u^2$, use the control $u = u^1$.
- 2) If $u^1 = u^2$, and $V_1 \geq V_2$, use the control $u = u^1$.
- 3) If $u^1 = u^2$, and $V_1 < V_2$, use the control $u = -u^1$.

This policy is shown graphically in Figure 7.3. Such a policy can be easily implemented on an analog computer to yield continuous closed-loop, time-optimal control. However, closed-loop, time-optimal control of second-order systems has been accomplished previously by other methods (Pontryagin 1962). Hence the primary importance of this result lies in the ability to generalize it to higher order systems in which $r = n-1$. Before doing this, however, it is perhaps of value to investigate in more detail the result just obtained.

So far the eigenvector scalar product technique of the preceding paragraph has only been established for the particular second-order system of Equation (7.10). In the following paragraphs, arguments are presented to show that this technique is valid for any second-order system with scalar control. The reader is reminded that only systems with real, distinct, and non-positive eigenvalues are being considered here. Emphasis is first placed on showing that for an arbitrary second-order system with scalar control,

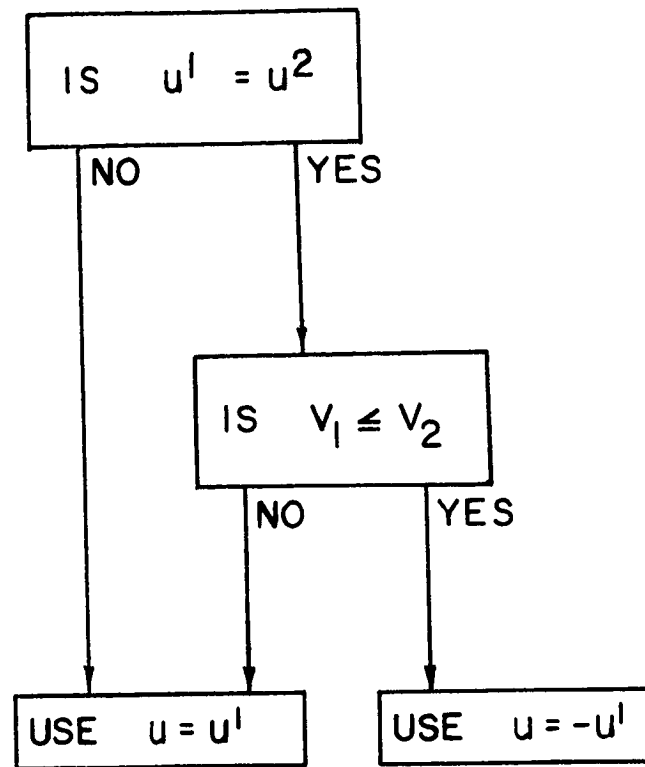


Figure 7.3 Sub-Optimal Policy for Second-Order Systems with Scalar Control

this technique transfers any initial state to R and thence to the origin. Next the fact that this technique yields, in fact, the time-optimal solution is proven.

In the following argument, the trajectory shown in Figure 7.2 is shown to be typical of all second-order systems of the class being considered here. Therefore, in order to add greater clarity to the argument, frequent reference is made to Figure 7.2. It must be emphasized that the trajectory shown follows from the argument which is independent of the graphical representation.

Consider an initial state, x^0 , arbitrarily located in Region I^- , i.e. $u^1(x^0) = u^2(x^0) = -1$ and $V_1(x^0) > V_2(x^0)$. Then by the use of the technique outlined above, the control u is set equal to $u^1(x^0)$. Since $u^1(x^0) = u^2(x^0)$, the value of both V_1 and V_2 must be decreasing as time increases, i.e. $V_1(x(t)) < V_1(x^0)$ for $t > 0$. Hence there are two possibilities to be considered.

First, the value of V_1 may reach zero before the value of V_2 does. But if this were to happen, then there is some intermediate state at which $V_1 = V_2$, since initially V_1 was greater than V_2 . But that state is in R , and hence the system has been transferred to R . However, V_1 cannot reach zero before V_2 , since at x^0 , $V_1(x^0) > V_2(x^0)$ and the value of $V_1(x^0)$ is equal to the time required to reach the $V_1 = 0$ surface. The main reason for presenting this argument here is that in the case of n th-order systems, it is

possible for the system to reach the R subset directly from Region I.

Second, the value of V_2 may become zero first. In this case u^2 must change sign after the $V_2 = 0$ line has been crossed. The system has therefore crossed into Region II. (See point x^a in Figure 7.2.) The control u , however, continues to be equal to $u^1 = -1$; since u^1 has not changed direction, the control is constant.

Since in Region II, $u = u^1 = -u^2$, the value of V_1 continues to decrease with time, while V_2 is now increasing. After a finite time, given by $V_1(x^a)$, the value of V_1 becomes zero. (See point x^b in Figure 7.2.) As the system crosses the $V_1 = 0$ line, u^1 changes sign to $+1$, and u^1 again becomes equal to u^2 , and u is now given by $-u^1$ and remains constant at -1 .

The system is now in Region III⁺ with $V_1 < V_2$ and $u^1 = u^2 = +1$. Since $u = -u^1$, the value of both V_1 and V_2 must be increasing with time. The question which arises is, "Would V_1 become greater than V_2 again and hence, at some time, is $V_1 = V_2$?"

Consider again the scalar functions, $Z_1(x)$, defined by

$$Z_1(x) = q^1 \cdot x \quad (7.21)$$

$i = 1, 2$

For the controlled case, the total time derivative of Z_1 becomes

$$\dot{Z}_1(x) = \lambda_1 Z_1(x) + q^1 \underline{B}u \quad (7.22)$$

Since the system is assumed to be controllable, $q^1 \underline{B}u$ must be non-zero for each i . Now consider the case where $\lambda_1 = 0$. Then \dot{Z}_1 becomes

$$\dot{Z}_1(x) = q^1 \underline{B}u \quad (7.23)$$

and $Z_1(x)$ must grow without bound as time increases. In case of a zero eigenvalue, the eigenvector scalar product solution as given by Equation (6.1) becomes

$$V(x, \rho) = \frac{|q^1 x|}{\rho \|\underline{B}^1 q\|} \quad (7.24)$$

Then by comparison of Equations (7.21) and (7.24), one sees that the value of V_1 is growing without bound. Because the eigenvalues are distinct, λ_2 must be negative and Z_2 is exponentially approaching a constant value. Therefore V_2 remains finite as time increases. Since V_1 is growing without bound while V_2 remains finite, it is obvious that at some time V_1 becomes equal to V_2 and the system reaches the R subset.

If both of the eigenvalues of \underline{A} are negative, then both Z_1 and Z_2 remain bounded and the system is driven toward a steady-state, x^{ss} , where $\dot{x} = 0$. Hence the value of both V_1 and V_2 does not increase without bound. This steady-state may be found by setting \dot{x} equal to zero in Equation (2.2) and then solving for x .

$$\underline{x}^{ss} = -\underline{A}^{-1}\underline{B}u \quad (7.25)$$

But $u = -1$, and hence Equation (7.25) becomes

$$\underline{x}^{ss} = \underline{A}^{-1}\underline{B}u \quad (7.26)$$

Now substituting \underline{x}^{ss} into Equation (5.28) for the eigenvector scalar product solution, one obtains

$$V_1(\underline{x}^{ss}) = \frac{1}{-\lambda_1} \ln \left(\frac{-\lambda_1 |\underline{q}^1 \cdot \underline{A}^{-1}\underline{B}|}{\|\underline{B}'\underline{q}^1\|} + 1 \right) \quad (7.27)$$

But \underline{q}^1 is an eigenvector of \underline{A}' , and hence

$$\underline{q}^1 \cdot \underline{A} = \lambda_1 \underline{q}^1 \cdot \quad (7.28)$$

Then post-multiplying both sides of Equation (7.28) by \underline{A}^{-1} gives

$$\underline{q}^1 \cdot = \lambda_1 \underline{q}^1 \cdot \underline{A}^{-1}$$

or

$$\underline{q}^1 \cdot \underline{A}^{-1} = \frac{1}{\lambda_1} \underline{q}^1 \cdot \quad (7.29)$$

Then substituting Equation (7.29) into Equation (7.27), one obtains

$$V_1(\underline{x}^{ss}) = \frac{1}{-\lambda_1} \ln \left(\frac{|\underline{q}^1 \cdot \underline{B}|}{\|\underline{B}'\underline{q}^1\|} + 1 \right) \quad (7.30)$$

Since $\underline{B}'\underline{q}^1$ is a scalar, $\|\underline{B}'\underline{q}^1\| = |\underline{B}'\underline{q}^1| = |\underline{q}^1 \cdot \underline{B}|$ and Equation (7.30) becomes

$$V_1(\underline{x}^{ss}) = \frac{1}{-\lambda_1} \ln(2) \quad (7.31)$$

In this case it can be readily seen that $V_1(x^{ss})$ is greater than $V_2(x^{ss})$, since $|\lambda_1| < |\lambda_2|$. Again there must be some state at which $V_1 = V_2$ and the system again reaches the R subset.

To review the process, from an arbitrary initial state in Region I^- , the system is transferred to Region II, then to III^+ , and finally it reaches R which is the boundary separating III^+ and I^+ . Since the initial point in Region I^- was arbitrary, for any initial state in Regions II and III^+ , the argument is identical from that point on. Similarly for initial states in Regions I^+ and III^- , the arguments are essentially identical to those presented above. Hence for any initial state the technique presented above transfers the system to the subset R. Once in R the control $u = u^1$ transfers the system directly to the origin, since $u^1 = u^2$ and $V_1 = V_2$.

It is easy to verify that the method presented above yields true time-optimal control. From an initial state, the method transfers the system by means of a constant control vector to the subset R. The control is then switched and the system is driven to the origin by means of another constant control vector. In the true time-optimal performance, the system is driven from an initial state to the switching line and then to the origin. But the switching line and the R subset are identical. The control necessary to transfer the system from any initial state to R is

unique due to the scalar nature of the control. Therefore the two methods must yield identical control in that portion of the response. But the control necessary to drive the system from the point on R to the origin is also unique, and hence the two methods must again yield the same control. The two methods are therefore identical for the entire transition period. The method presented above yields true time-optimal performance.

Before generalizing the above procedure to n th-order systems in which $r = n-1$, it is convenient to add some new notation. Let $u^{2,n}(x^0,1)$ be the vector of unit magnitude such that $V_2(x^0, \rho_2) = V_3(x^0, \rho_3) = \dots = V_n(x^0, \rho_n)$ where as usual $\rho_1 = u^{2,n}(x^0,1) \cdot u^1(x^0,1)$. Then let the common value of these r eigenvector scalar product solutions be designated by $V_{2,n}(x^0,1) = V_2(x^0, \rho_2) = \dots = V_n(x^0, \rho_n)$.

One additional generalization is necessary. Since the control is no longer scalar, the concept of equating two controls needs to be generalized. The approach that is taken is to consider the sign of the scalar product of two control vectors. Therefore in the second-order case, one should consider $u^1 u^2 > 0$ rather than $u^1 = u^2$ in order to be completely consistent.

Again let the state space be divided into three regions.

Region I: $u^1(x,1) \cdot u^{2,n}(x,1) > 0$ and $V_1(x, \rho_1) > V_{2,n}(x,1)$.

Region II: $u^1(x,1) \cdot u^{2,n}(x,1) < 0$.

Region III: $u^1(x,1) \cdot u^{2,n}(x,1) > 0$ and $V_1(x, \rho_1) < V_{2,n}(x,1)$.

Regions I and III are further subdivided into positive and negative regions, depending on whether $q^1 \cdot x$ is positive or negative.

Now generalizing the control policy for second-order systems presented above, one obtains:

- 1) If $u^1 \cdot u^{2,n} < 0$, use $u = u^1$.
- 2) If $u^1 \cdot u^{2,n} > 0$ and $V_1 > V_{2,n}$, use $u = u^1$.
- 3) If $u^1 \cdot u^{2,n} > 0$ and $V_1 < V_{2,n}$, use $u = -u^1$.
- 4) If $u^1 \cdot u^{2,n} > 0$ and $V_1 = V_{2,n}$, use $u = u^{2,n}$.

This policy is shown graphically in Figure 7.4. Again the policy can be implemented on an analog computer to yield continuous closed-loop, sub-optimal control. The sub-optimal policy is based on two facts. First, the technique reduces to true time-optimal control for second-order systems as shown above. Second, the method works, i.e. it transfers the system from an initial state to the R subset and thence to the origin. This second fact can be proven by an argument similar to that presented above for second-order systems. Because of this similarity, the argument is not repeated here. However, it is necessary to show that once the system enters Region III, it is driven again into

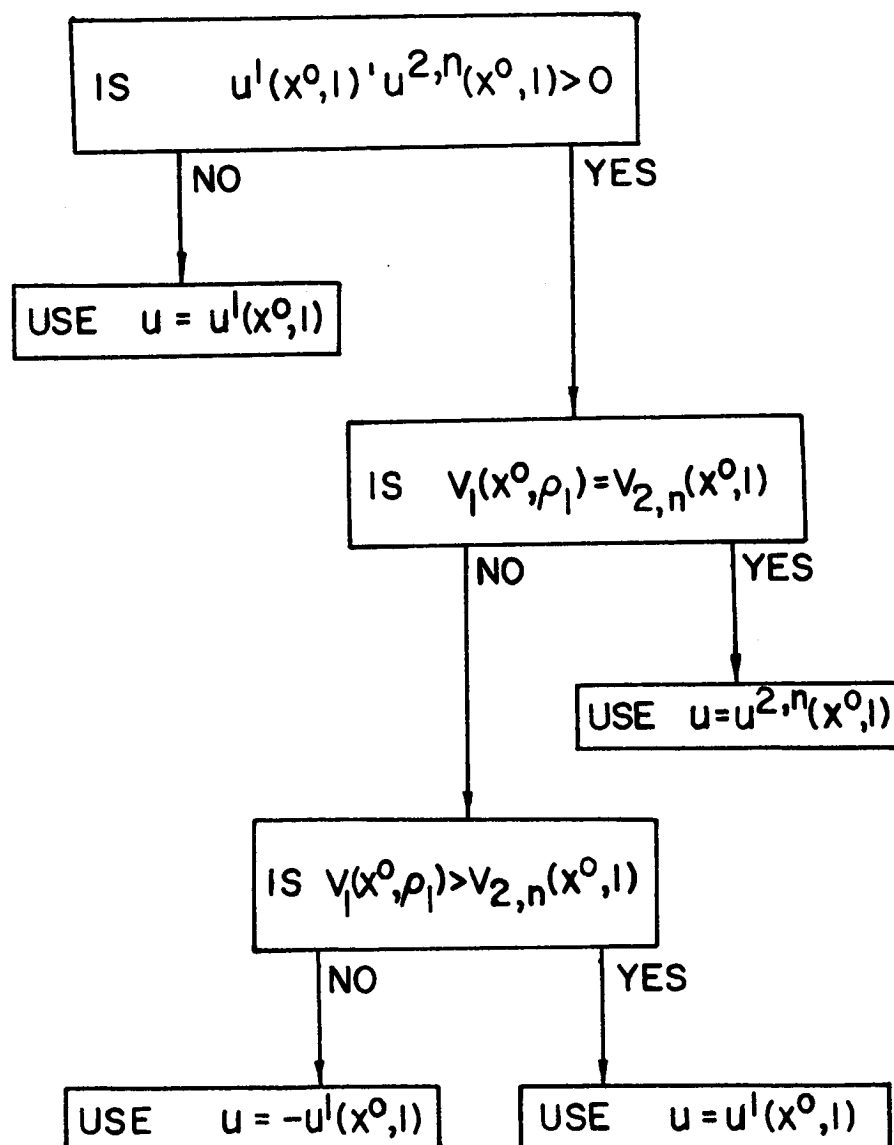


Figure 7.4 Sub-Optimal Policy for
n-th Order Systems with $r = n-1$

Region I and thus must cross the R subset which separates Regions I and III.

In the case where $\lambda_1 = 0$, it is easy to show by an argument similar to that for second-order systems that V_1 becomes unbounded, while $V_{2,n}$ remains finite. Hence from Region III the system is driven into Region I and therefore reaches the R subset. Unfortunately in the case where λ_1 is negative, it is not possible to show in general that the system reaches the R subset, since it is impossible to obtain a general expression for $V_{2,n}(x^{ss}, 1)$. Because the argument for $\lambda_1 = 0$ carries over so directly, it seems reasonable to assume that the case for $\lambda_1 \neq 0$ is also true. However, one should check this assumption for any particular systems of interest.

As an illustration of the above technique, consider the following third-order system with two control variables.

Example 7.3 The laws of motion for the system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (7.32)$$

It is desired to drive the system from the initial state $(1.0, -1.0, -2.0)$ to the origin in approximately minimum time. It can be easily verified that $V_1(x^{ss}, \rho_1) > V_{2,n}(x^{ss}, 1)$ and hence the procedure described above can be applied.

Figure 7.5 shows the time response of this system when subjected to the above sub-optimal procedure. The system begins in Region I⁻ and the control is therefore $u = u^1(x^0, 1) = (0, 1)$. At approximately $t = 0.9$, the system crosses into Region II and the control remains equal to $(0, 1)$. The system enters Region III⁺ at approximately $t = 1.1$, while the control remains constant. The R subset is reached at $t = 1.8$ and the control is switched to $u^{2,n}(x, 1) = (-.285, -.959)$. The system reaches the origin at $t = 2.23$.

Hence by the above sub-optimal procedure, the system has been transferred from an initial state to the origin. Although this example is presented from a specific initial state, the policy can be implemented on an analog computer to provide continuous closed-loop operation.

If $r < n-1$, then the method presented above cannot be applied directly. In this case it is necessary for the system to pass through a sequence of subsets with ever-decreasing dimensionality, until the r -dimensional subset, R , is reached. Each member of this sequence of subsets must satisfy two requirements: 1) it must contain all of the following subsets, and 2) it must be possible for the system to remain in the subset under a constant control.

One possible sequence is $V_{n-r} = V_{1+n-r, n}$, $V_{n-r-1} = V_{n-r, n}$, ..., $V_1 = V_{2, n}$. Although this sequence does satisfy the first requirement, it is easy to show, by example, that

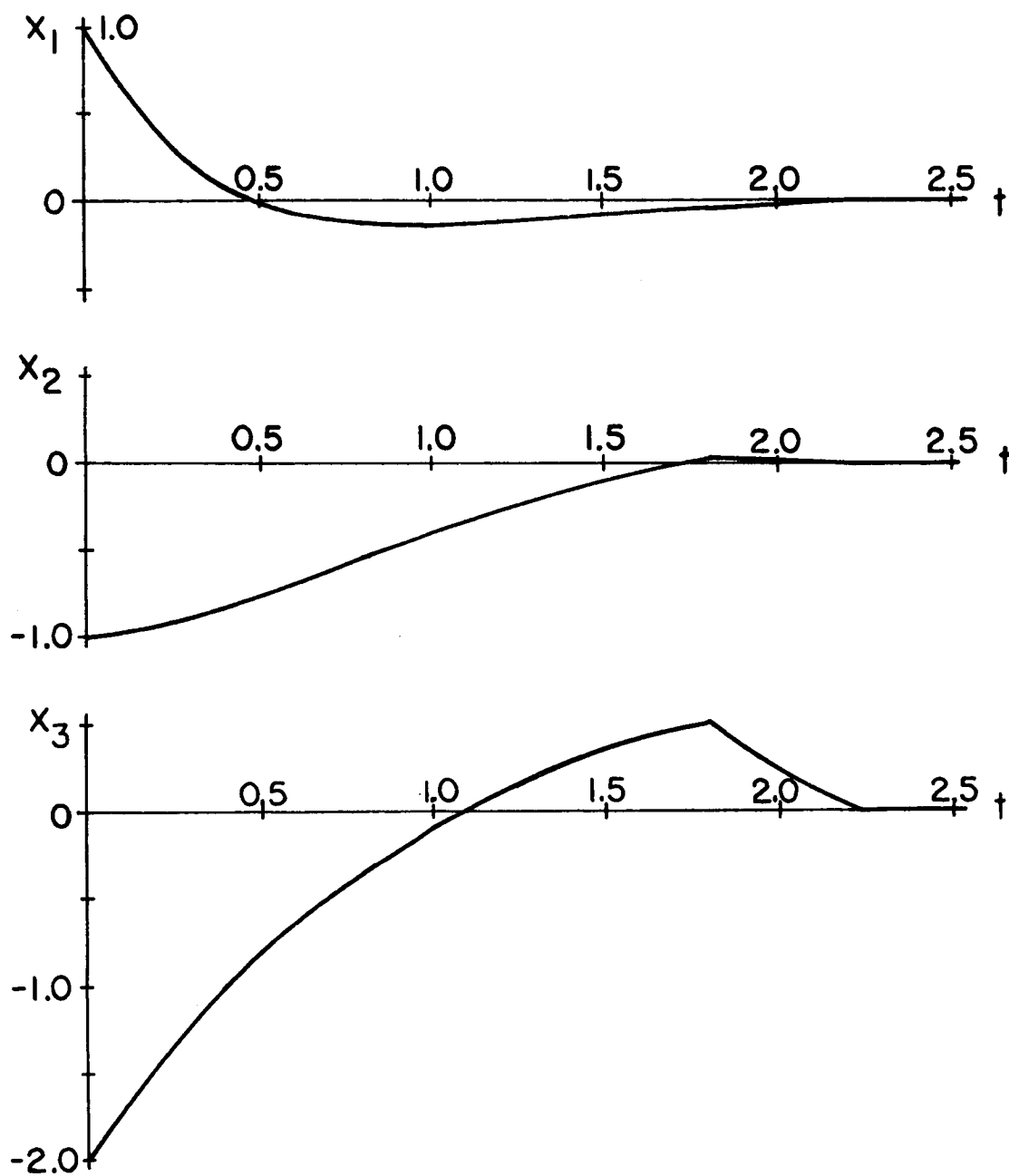


Figure 7.5 Time Response of a
Third-Order System with Two Control Variables

it does not satisfy the second. It does not seem possible to find a simple description for such a sequence in terms of the eigenvector scalar products.

Therefore, although the method presented in this section would appear to be the most promising in terms of a satisfactory compromise between system complexity and speed of response, its use at present is limited to systems in which $r = n-1$.

In the next section the method of this section is combined with the method of Section 7.2 to gain additional flexibility.

7.5 Combination of Methods

Two methods of sub-optimal control have been developed in this chapter for systems in which the control matrix is singular.

The first method, presented in Section 7.2, is based on the concept of equating r of the eigenvector scalar product solutions. The system is therefore driven from an arbitrary initial state to a state where r of the eigenvector scalar product solutions are zero. The control is turned off at that point and the system is allowed to approach the origin exponentially with a time constant equal to $1/|\lambda_{r+1}|$. This sub-optimal method can be most advantageously applied in cases where r or less of the eigenvalues are small relative to the remaining eigenvalues. It was

further noted that the approach presents a good compromise between time- and fuel-optimal performance.

The second method, presented in Section 7.4, is based on the concept of determining the r -dimensional subset of the state space where all n of the eigenvector scalar product solutions can be equated. Because this subset is only r -dimensional, an arbitrary initial state is not in R and hence it is necessary to transfer the system to R . Unfortunately, it is only possible to determine a method for doing this in the case where $r = n-1$. However, for systems in which $r = n-1$, it is shown that the origin can be reached in a finite time by the use of two different control vectors with a change of control taking place upon reaching the R subset. It is also shown that for second-order systems with scalar control that this method yields true time-optimal performance.

As noted above, each of the two methods has limitations as well as advantages. However, the two methods can be used in combination in order to eliminate some of these limitations. As an illustration of the combination of these methods, consider the following third-order system with scalar control.

Example 7.4 The laws of motion for the system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

The eigenvalues are found to be $-1, -2, -10$. Neither of the above methods can be applied separately to yield a satisfactory answer. An application of the second method is impossible since $r \neq n-1$. On the other hand, the first method cannot be employed successfully either, since the system approaches the origin with a time-constant of $1/2$ after the control is terminated. However, by combination of these methods, a very satisfactory answer can be obtained.

The second method can be used to first drive the system to a state where $V_1 = V_2$. Then applying the first method, the system is driven to a state where $V_1 = V_2 = 0$, thus eliminating the two longer time-constants. The system therefore approaches the origin with a time-constant of $1/10$.

This sub-optimal method was implemented on an analog computer, and the time response for an initial state $x^0 = (1.3, -0.75, 0)$ is shown in Figure 7.6. The scalar functions, $Z_1(x) = q^1 \cdot x$, are plotted rather than the state variables, since they show more distinctly the point at which $V_1 = V_2 = 0$.

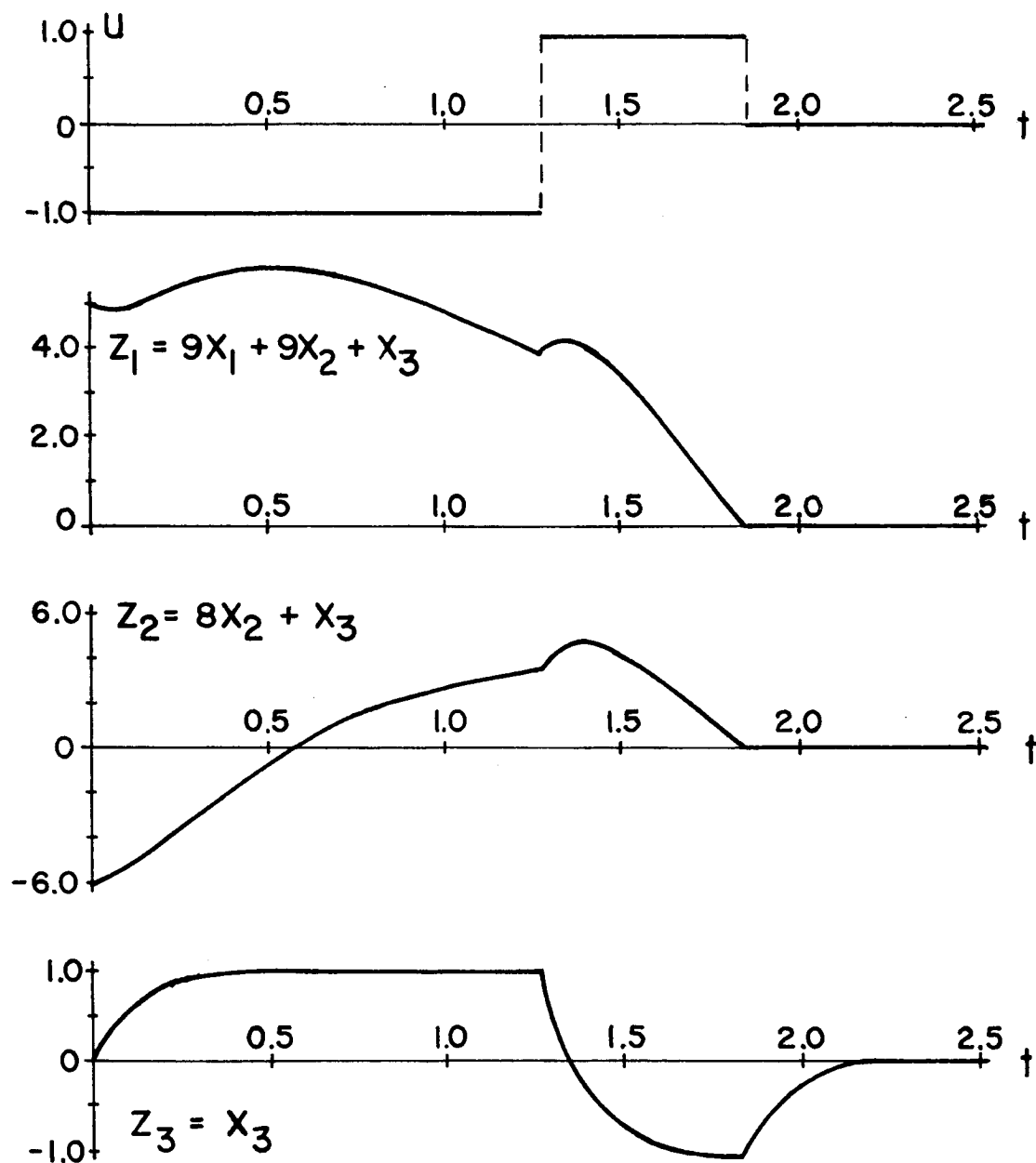


Figure 7.6 Time Response of a
Third-Order System with Scalar Control

At $t = 1.28$ the state $V_1 = V_2$ is reached and the control switches from -1 to $+1$ and the system is driven toward the state $V_1 = V_2 = 0$. This state is reached at $t = 1.84$, and the control is turned off. Since the two slow time constants have been eliminated during the controlled portion, the system approaches the origin with only the time constant $1/10$ present.

This sub-optimal method has several advantages. First, the response time is probably not too much longer than the absolute minimum. It appears reasonable to assume that the minimum time cannot be less than the time required to force $V_1 = V_2 = 0$, 1.84 seconds. Second, the method provides continuous, simple, closed-loop operation. These aspects cannot be over-emphasized. The method also conserves fuel during the final coast period. Again an effective compromise between speed of response and system complexity has been reached.

Unfortunately, the S_2 -type surfaces cannot be used to judge the quality of the sub-optimal control for either of the methods presented in this chapter. In the case of the Bang-Coast method, the response time is infinite, since the origin is approached exponentially. The only possibility is to define some finite region about the origin and find the time required to reach it. In this form it is almost impossible to find the sub-optimal isochrones and

hence comparing them to the S_2 surfaces is difficult, if not impossible.

In the case of the method of Section 7.4, the sub-optimal isochrones are difficult to find except in the case of second-order systems. Unfortunately, in this case, it is also easy to find the true time-optimal isochrones which are actually the same as the sub-optimal ones. However, the S_2 surfaces of Section 5.4 do not closely approximate the S_0 surfaces, as can be easily seen from a simple example. Because of this lack of close approximation for second-order systems, it does not appear that comparing the sub-optimal and S_2 surfaces offers much assistance for higher order systems.

The two methods presented in this chapter for the control of systems with singular control matrices are by no means a final solution to the problem. A great deal of additional research is still needed, particularly with respect to the method presented in Section 7.4. In the next chapter, a procedure is developed for applying the methods of this and the preceding chapter to some practical control problems.

CHAPTER 8

APPLICATIONS TO NUCLEAR ROCKET CONTROL PROBLEMS

8.1 Introduction

In this chapter, a procedure is outlined for applying the closed-loop, approximately time-optimal control methods of the two preceding chapters to some practical control problems. In particular, attention is directed toward problems of nuclear rocket control. It should be emphasized that the intent is to show how the methods can be applied to typical problems rather than to actually apply them to problems of practical significance.

One encounters three obstacles in attempting to apply the previously developed methods to practical control problems:

- 1) The desired terminal state is not the origin.
- 2) The differential equations describing the system are generally nonlinear.
- 3) The control vector is not constrained in the norm.

Since each of these three factors violates the basic assumptions of the sub-optimal control methods, it is necessary that the problems be modified before these methods can be applied. Appropriate modifications in order to overcome each of these three factors are presented in the following development.

The difficulty of a terminal state, $x(T)$, which is not the origin may be overcome by making a transformation of coordinates. If a new set of state variables, y_1, y_2, \dots, y_n , is defined by the relation

$$y_i = x_i - x_i(T) \quad i = 1, 2, \dots, n \quad (8.1)$$

then the terminal state is the origin in terms of the y state variables. It should be noted that if the terminal state is not an equilibrium state, then some transformation of the control vector may also be necessary. (See Section 8.3.)

Having transformed the terminal state to the origin, one may satisfy the requirement of a linear system by linearizing the system equations about the origin. In order to be able to apply the sub-optimal methods to this linearized version of the system, it is necessary that the eigenvalues be real, distinct and non-positive, thereby limiting to some extent the class of systems which can be treated. The effectiveness of the sub-optimal control methods depends, to a large measure, on the validity of this linearization procedure. If, for example, most initial states lie within the region for which the linearization is accurate, then good performance can be expected. If, on the other hand, many initial states are outside the region of validity, then the quality of the performance may be in question. However, due to the closed-loop nature of the system, as long as the

linearization is even "reasonably" accurate, the system should be rapidly driven toward the region of validity, and hence the performance may still be good. Once again closed-loop control is shown to be highly desirable.

The fact that in many cases the control vector is not constrained in the norm is perhaps the most difficult obstacle to overcome in applying the sub-optimal methods to practical control problems. The method of attack is to find an approximation to the given control region in the form of a constraint on the norm of the control vector. In other words, even though the control vector is not initially constrained in the norm, a norm constraint is generated and employed. In order to make this approximation most conveniently, consider a form of the generalized norm mentioned in Section 2.3, $\|\underline{D}u\|^2 \leq 1$. The problem is then to pick the elements of \underline{D} in order to generate the best approximation to the given control region. Having chosen \underline{D} , then if a new set of control variables is defined by $w = \underline{D}u$, the control region becomes $\|w\|^2 \leq 1$, and the problem has been put in proper form for the application of the sub-optimal control methods.

As an illustration of this procedure, consider the second-order control region defined by

$$|u_1| \leq 4 \qquad |u_2| \leq 1$$

(See Figure 8.1.) In this case, \underline{D} might be selected as

$$\underline{D} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix}$$

in which case the approximate control region is $\|\underline{D}u\|^2 \leq 1$, shown as the cross-hatched area in Figure 8.1.

To summarize, the procedure to be followed in preparing a practical control problem for the application of the previously developed sub-optimal control methods is

- 1) Transform the desired terminal state to the origin.
- 2) Linearize the system about the origin.
- 3) Approximate the control region with a norm constraint.

In the following sections, the above procedure is illustrated by application to two simple nuclear rocket control problems. Once again it should be emphasized that the intent is to show how to apply, not to apply, the procedure to practical control problems.

8.2 Time-Optimal Control of Single-Precursor Group Neutron Kinetics

In this section, the above procedure is applied to the time-optimal control of single-precursor group neutron kinetics. The system is defined by the differential equations (See Mohler 1965, page 35.)

$$\begin{aligned} \dot{n} &= -\frac{\beta}{\ell} n + \lambda c + \frac{\beta}{\ell} n u \\ \dot{c} &= \frac{\beta}{\ell} n - \lambda c \end{aligned} \tag{8.2}$$

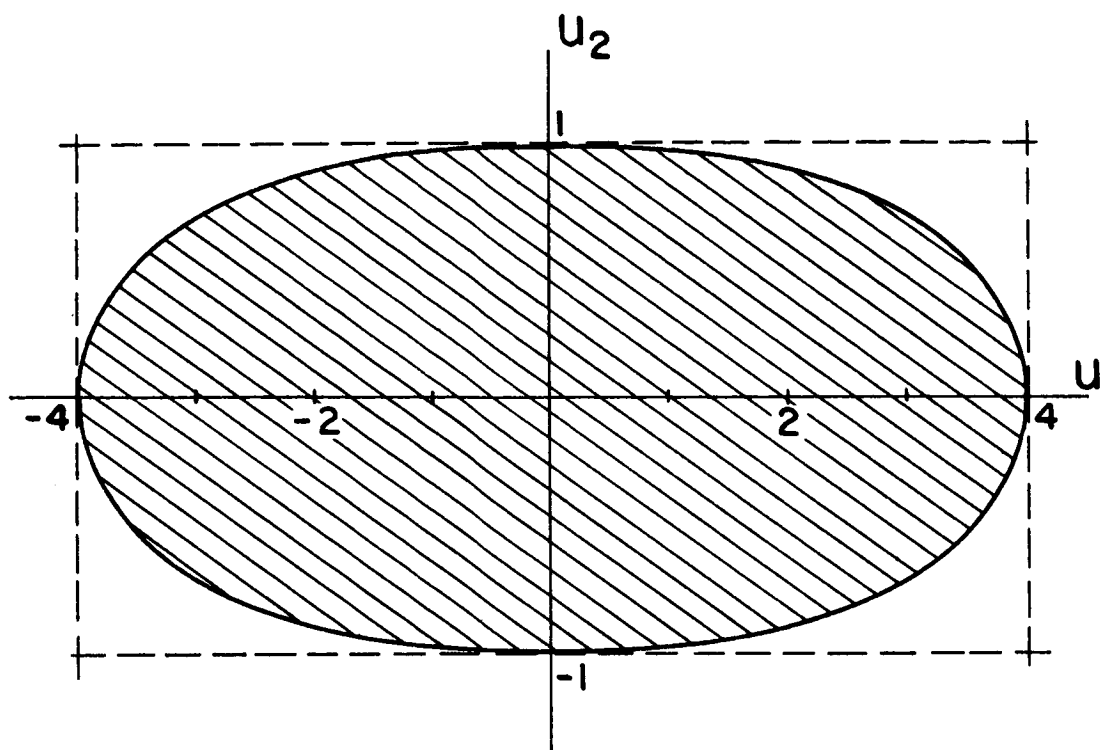


Figure 8.1 Approximation of
a Typical Control Region

where n is the neutron density, c is the precursor concentration, and u is the reactivity control. The problem is to develop a closed-loop controller which transfers the system from any initial state to a desired terminal state $n(T)$ and $c(T)$ in approximately minimal time. The control region is given by $|u| \leq 0.5$.

In order to treat the problem numerically, the following parameters were used:

$$\begin{aligned}\lambda &= 0.1 \text{ sec}^{-1} \\ \beta &= 6.0 \times 10^{-3} \\ \lambda &= 3 \times 10^{-5} \text{ sec}\end{aligned}$$

in which case the system equations (8.2) become

$$\begin{aligned}\dot{n} &= -200n + 0.1c + 200nu \\ \dot{c} &= 200n - 0.1c\end{aligned}\tag{8.3}$$

The terminal state was selected as

$$\begin{aligned}n(T) &= 1.5 \\ c(T) &= 3 \times 10^3\end{aligned}$$

Now applying the procedure outlined above, the first step is to transform the desired terminal state to the origin, in which case the system equations (8.3) become

$$\begin{aligned}\dot{y}_1 &= -200y_1 + 0.1y_2 + 200(y_1 + 1.5)u \\ \dot{y}_2 &= 200y_1 - 0.1y_2\end{aligned}\tag{8.4}$$

where

$$y_1 = n - 1.5$$

$$y_2 = c - 3 \times 10^3$$

Linearizing the system, one obtains

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -200 & 100 \\ 200 & -0.1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 300 \\ 0 \end{bmatrix} u \quad (8.5)$$

The eigenvalues of the linearized system are 0, -200.1 and therefore the requirement for real, distinct and non-positive eigenvalues is satisfied.

Since the control is scalar, $\|u\| = |u|$, and the control may be put into the correct form by making the substitution $w = 2u$, in which case the control region becomes

$$|w| = \|w\| \leq 1 \quad (8.6)$$

and the system equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -200 & 100 \\ 200 & -0.1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 150 \\ 0 \end{bmatrix} w \quad (8.7)$$

Since $r = n-1$, either the Switched Control or the Bang-Coast Method may be employed. In order to achieve finite settling time, the Switched Control Method was selected, and the closed-loop control system was implemented on a digital computer. For an initial state

$$n = 1.0$$

$$c = 2.0 \times 10^3$$

the optimal transition time is 4.084 seconds, while the sub-optimal is 4.086, for a difference of less than 0.1%. It cannot be over-emphasized that the sub-optimal system is closed-loop and correspondingly yields good results for any initial state. Although this problem is perhaps the exception rather than the rule as concerns the closeness of optimal and sub-optimal performance, the approach produces a relatively simple, closed-loop system in a systematic manner.

8.3 Time-Optimal Control of a Simple Nuclear Rocket Engine

As a second example of the procedure outlined in the first section, consider the time-optimal control of a simple nuclear rocket engine. The model (Mohler and Perry 1964) for this problem consists of the basic neutron kinetic equations coupled with a heat exchange equation by means of core temperature and propellant flow rate in the form of reactivity. For convenience, only one group of delayed neutrons is used. The system equations are

$$\begin{aligned}\dot{Q} &= \frac{C_T}{\ell} TQ - \frac{\beta}{\ell} Q + \lambda C + \frac{\beta}{\ell} Qu_1 + \frac{C_P}{\ell \sqrt{T}} Qu_2 \\ \dot{C} &= \frac{\beta}{\ell} Q - \lambda C \\ \dot{T} &= \frac{Q}{M_c} - \text{Tau}_2\end{aligned}\tag{8.8}$$

where the state variables are

Q = power level, megawatts

C = precursor concentration

T = core temperature, $^{\circ}\text{R}$

and the control variables are

u_1 = control rod reactivity

u_2 = coolant mass flow rate

Due to turbo-pump design constraints, such as stalling and cavitation, both upper and lower bounds are placed on u_2 . In addition, there is a constraint on the maximum and minimum control rod reactivity. Hence the control region becomes

$$-K \leq u_1 \leq K$$

$$u_a \leq u_2 \leq u_b \quad (8.9)$$

The problem is to obtain a closed-loop control system which transfers the system from any initial state to the terminal state, Q_0 , C_0 , and T_0 , in approximately minimum time.

In order for the system to remain at the terminal state, the flow must be equal to

$$u_{20} = \frac{Q_0}{M_c T_0 a} \quad (8.10)$$

and hence it is necessary to transform not only the state variables but also the control variables. If this desired terminal state is transformed to the origin and the system is linearized, the equations of motion become

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} (C_T T_o + \frac{C_P U_{2o}}{\sqrt{T_o}} - \beta)/\ell & \lambda & (C_T Q_o - \frac{C_P U_{2o} Q_o}{2\sqrt{T_o}^3})/\ell \\ \beta/\ell & -\lambda & 0 \\ \frac{1}{M_c} & 0 & -a u_{2o} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{Q_o \beta}{\ell} & \frac{C_P Q_o}{\ell \sqrt{T_o}} \\ 0 & 0 \\ 0 & -a T_o \end{bmatrix} \begin{bmatrix} u_1 \\ u'_2 \end{bmatrix} \quad (8.11)$$

where

$$y_1 = Q - Q_o$$

$$y_2 = C - C_o$$

$$y_3 = T - T_o$$

$$u'_2 = u_2 - u_{2o}$$

The control region is then

$$-K \leq u_1 \leq K$$

$$u_a - u_{2o} \leq u'_2 \leq u_b - u_{2o} \quad (8.12)$$

In order to treat the problem numerically, the following hypothetical parameters were chosen

$$C_T = -2.4 \times 10^{-6} \text{ } ^\circ\text{R}^{-1}$$

$$C_P = 2.31 \times 10^{-3} \text{ } ^\circ\text{R-sec/lb}$$

$$a = 5.22 \times 10^{-3} \text{ lb}^{-1}$$

$$\beta = 6 \times 10^{-3}$$

$$\ell = 3 \times 10^{-5}$$

$$M_c = 1.2 \text{ megawatts}$$

$$\lambda = 0.1 \text{ sec}^{-1}$$

$$K = 1.5$$

$$u_a = 30$$

$$u_b = 250$$

$$Q_o = 2 \times 10^3$$

$$C_o = 4 \times 10^6$$

$$T_o = 2.5 \times 10^3$$

$$u_{2o} = 130$$

in which case the system equations become

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -200 & 0.1 & -240 \\ 200 & -0.1 & 0 \\ 0.833 & 0 & -0.667 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 4 \times 10^5 & 3077 \\ 0 & 0 \\ 0 & -12.82 \end{bmatrix} \begin{bmatrix} u_1 \\ u'_2 \end{bmatrix} \quad (8.13)$$

The eigenvalues for this linearized version of the system are -0.0623, -1.612, and -199.09 and the requirement for real, distinct and non-positive eigenvalues is satisfied.

This completes the first two steps in the procedure of Section 8.1 and leaves only the approximation of the

control region by means of a norm constraint. Since the control region has become

$$-1.5 \leq u_1 \leq 1.5$$

$$-100 \leq u'_2 \leq 120$$

one possible \underline{D} matrix is

$$\underline{D} = \begin{bmatrix} \frac{1}{1.5} & 0 \\ 0 & \frac{1}{100} \end{bmatrix}$$

It should be noted that, in this case, it is not possible to reach the upper boundary of u_2 with the norm constraint due to the unsymmetrical nature of the control region. (See Figure 8.2.) Defining a new set of control variables $w = \underline{D}u$, the system equations become

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -200 & 0.1 & -240 \\ 200 & -0.1 & 0 \\ 0.833 & 0 & -0.667 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 6 \times 10^5 & 3.077 \times 10^5 \\ 0 & 0 \\ 0 & -1282 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (8.14)$$

and the control region is given by $\|w\| \leq 1$.

Once again $r = n-1$ and either of the methods of Chapter 7 can be used for the design of the sub-optimal controller, although the Switched Control Method is probably more desirable. A straight-forward application of this method yields a closed-loop sub-optimal system. The degree

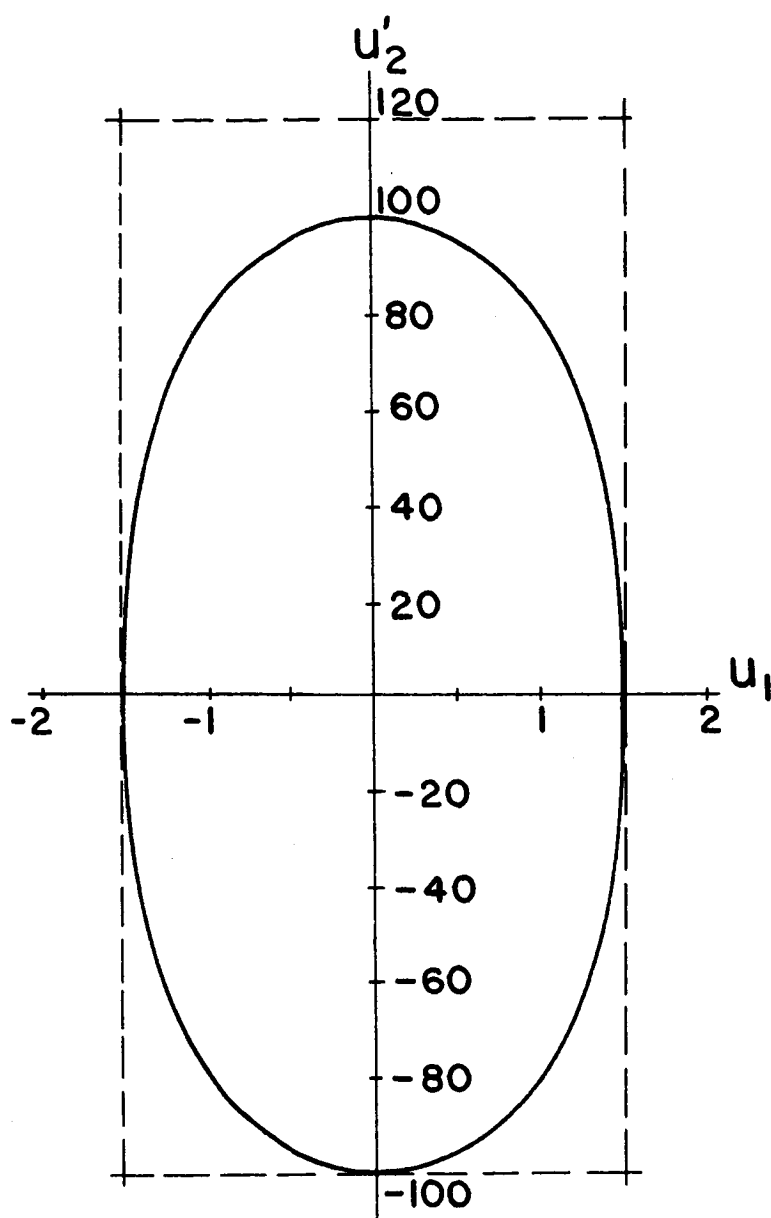


Figure 8.2 Approximation of the Control Region by a Norm Constraint

to which the performance of the sub-optimal system approximates the optimal is impossible to determine except by determining the optimum response for specific trajectories.

8.4 Summary

In this chapter a procedure is outlined for applying the sub-optimal methods of Chapters 6 and 7 to practical control systems. This procedure is illustrated by its application to two simple nuclear rocket control problems. It is perhaps of value to summarize briefly the advantages and disadvantages of the procedure.

Since a closed-loop system is obtained, any initial state is transferred to the desired terminal state as long as the linearization is reasonably accurate. The procedure is systematic and relatively easy to employ. Since the methods of Chapters 6 and 7 can only be used for time-optimal control of linear systems with a control norm constraint, all problems must be forced into this rather restrictive mold.

The success of this process for any particular problem depends on several factors and hence is difficult to predict. In particular, the degree of validity of the linearization and the closeness to which the admissible control region can be approximated in Step Three are important factors in determining the success or failure of the method. The degree to which the sub-optimal performance

approximates the true optimal is also unknown and hence one must simply decide if the sub-optimal system is adequate for any given problem.

The next chapter presents a summary of this report and several suggestions for further research.

CHAPTER 9

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

9.1 Summary

In this work the Second Method of Liapunov is combined with the minimum principle to develop a method of closed-loop, approximately time-optimal control of linear systems whose laws of motion are described by

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}u$$

The eigenvalues of \underline{A} are real, distinct, and non-positive. The system is to be transferred from an arbitrary initial state, \underline{x}^0 , to the origin. The control region, U , is the set of all control vectors, u , such that $\|u\| \leq 1$. The first step in this development is to show that solving the basic optimization problem is equivalent to solving a first-order partial differential equation, the Hamilton-Jacobi equation. Although it is not possible to solve this equation in general, a special class of solutions is shown to exist. These solutions, called eigenvector scalar products, form the basis of an effective closed-loop, sub-optimal control method.

In Chapter 6 the eigenvector scalar product solutions are used for the control of systems in which the control matrix, \underline{B} , is non-singular. This method is based on

the concept of finding a control vector of unit magnitude such that the time required to reach each of the n surfaces of $V_1 = 0$ is equal. Such a control vector can be found by solving the n simultaneous algebraic equations $V_1(x^0, \rho_1) = V_2(x^0, \rho_2) = \dots = V_n(x^0, \rho_n)$ and $\|u\| = 1$ where $\rho_1 = u^1 u^1(x^0, 1)$. This constant control vector transfers the system to the origin in a finite time.

This method has several advantageous features. First, in the absence of disturbances, the control vector, once obtained, remains constant until the system reaches the origin. Second, the transition time from any point to the origin is finite and can be readily obtained. The sub-optimal isochrones can also be easily found. Third, in designing a closed-loop system using this method, the controller-computer must only solve algebraic equations, and hence the control can be computed continuously. This should be contrasted with many of the present methods which require on-line solution of two-point boundary value problems, and hence discrete control.

Although obtaining the optimal isochrones is computationally impossible, by the use of the eigenvector scalar product solutions, it is possible to find surfaces which bound the optimal isochrones from the outside. These bounds on the optimal isochrones enable one to show that the sub-optimal and optimal isochrones are tangent at $2n$ points. Because of the closeness of these surfaces at $2n$ points,

it appears reasonable to conclude that the performance of the sub-optimal system is an acceptable compromise between system complexity and speed of response. The inability to find bounding surfaces has been a serious difficulty in designing approximately time-optimal systems in the past.

In Chapter 7 the restriction of a non-singular control matrix is removed. The result is that it is no longer possible to develop a single method to handle all problems. Hence, two methods are developed, each of which has special restrictions.

The first method, called the Bang-Coast Method, is based on the concept of finding a control vector such that the time required to reach r of the $V_1 = 0$ surfaces is equal where r is the number of control variables. Such a control transfers the system from any initial state to a state where r of the eigenvector scalar product solutions are zero. At this point the control is turned off and the system is allowed to coast uncontrolled toward the origin. If the magnitudes of r or less of the eigenvalues are small relative to the remaining eigenvalues, then by proper choice of the eigenvector scalar product solutions the small eigenvalues can be effectively eliminated from the system during the control-led interval. The system then approaches the origin with only the relatively fast time constants present.

The second method, called the Switched Control Method, is based on the concept of determining the

r -dimensional subset of the state space where all n of the eigenvector scalar product solutions can be equated. Because this subset is only r -dimensional, an arbitrary initial state is not in R and hence it is necessary to transfer the system to R . Unfortunately, it is only possible to determine a method for doing this in the case where $r = n-1$. However, for systems in which $r = n-1$, it is shown that the origin can be reached in a finite time by the use of two different control vectors with a change of control taking place upon reaching the R subset. It is also shown that for second-order systems with scalar control that this method yields true time-optimal performance.

It is obvious that each of the two methods has limitations as well as advantages. By combining the methods, some of the limitations can be eliminated. In this manner effective sub-optimal control of systems with singular control matrices can be obtained. As before, both of these methods can be implemented on an analog computer to achieve continuous closed-loop operation; this fact is verified by two examples.

These two methods, however, can by no means be considered as a final solution to the problem. A great deal of additional research is still needed, particularly with respect to the Switched Control Method.

In Chapter 8, a procedure is outlined for applying the sub-optimal control methods developed in Chapters 6 and

7 to practical control problems. This procedure is illustrated by means of application to two simple nuclear rocket problems.

In summary, there are three major contributions of this work. First, a special class of solutions of the Hamilton-Jacobi equation, called eigenvector scalar products, are shown to exist.

Second, a method of judging the quality of a sub-optimal system by bounding the optimal isochrones is developed. The eigenvector scalar product solutions are used to bound the optimal isochrones in an unusually simple manner. Although this method of bounding the optimal isochrones still needs much refinement, it is an important first step and should allow at least a gross estimation of the quality of sub-optimal systems.

Third, the eigenvector scalar product solutions are used to develop several methods of sub-optimal control. The procedure to be followed in each method is systematically presented, and in each case the method is shown to represent an effective compromise between system complexity and speed of response.

The systematic procedures of this work should be contrasted with the arbitrary selection of a V-function required by the methods outlined in the Appendix. In addition, the methods presented in the Appendix use quadratic forms for V-functions, and hence the "best" V-function

depends on the initial state of the system. In the case of the Kalman-Bertram method, chattering near the origin degrades the performance considerably. Nahi's method, on the other hand, requires that \underline{B} be non-singular.

9.2 Suggestions for Future Research

Although the sub-optimal method presented in this work offers an effective compromise between system complexity and speed of response for many systems, there are several extensions of the method which would greatly increase the number of systems to which it applies.

Although a method of treating nonlinear systems by means of linearization about the terminal state was outlined, it would be of significant practical value to be able to extend the sub-optimal methods directly to nonlinear problems. The most encouraging area at present is bilinear systems, in which the state and control variables are separately linear but jointly non-linear. Because of their close relation to linear systems, it appears quite possible that the method can be successfully applied to bilinear systems. In regard to non-linear systems, the technique of canonic transformation, presented briefly in Section 5.2, may prove to be very valuable. Additional investigation of this area would be highly desirable.

An extension of the method to cases where $L(x) \neq 1$ would also greatly increase the practical importance of the

method. Again a different method of modifying the Hamilton-Jacobi equation may prove valuable. Of particular importance would be the case where $L(x) = x'Px$ and the problem becomes the minimization of a quadratic performance index.

A method of refining the bounds on the optimal isochrones as obtained in Section 5.4 would increase the usefulness of these S_2 surfaces. The importance of good S_2 surfaces with regard to judging the quality of a sub-optimal system cannot be over-emphasized. Unless the optimal isochrones can be obtained, in which case a sub-optimal method is usually not needed, the S_2 surfaces provide the only approach to judging sub-optimal systems.

Additional research is needed with regard to the Switched Control Method of Chapter 7. In particular, methods of reaching the R subset where $r \neq n-1$ are needed. If the method could even be extended to the case where $r = n-2$, it would be of great practical significance.

With regard to applications to practical systems, methods of treating control constraints other than a norm constraint and methods of handling state space constraints would be valuable. Removing the requirement of real eigenvalues should also be investigated in order to increase the range of practical application.

By means of such extensions, it should be possible to obtain closed-loop, sub-optimal methods of great practical significance.

APPENDIX A

SUB-OPTIMAL CONTROL METHODS USING THE SECOND METHOD

A.1 Introduction

In this appendix, several methods of designing sub-optimal control systems by the use of the Second Method of Liapunov are presented. The methods presented here are not intended to be an exhaustive compilation of such methods but rather were chosen because of their relation to the material in Chapter 4.

Each of the following three sections begins with a brief discussion of the concepts or ideas underlying that method. This is followed by a short presentation of the method, which is then illustrated by a numerical example. The sections conclude with a discussion of the advantages and disadvantages of each method. For each of the methods presented, the uncontrolled system is assumed to be at least stable in the sense of Liapunov.

Unfortunately, all of these methods have three basic problems: (1) they are approximate, (2) either no estimate of the approximation error is possible, or the estimate is overly conservative, and (3) it is necessary to choose a $V(x)$ for which no general procedure is presented. Hence these methods have not been widely accepted.

A.2 Estimation of Transient Behavior

One of the first uses of the Second Method as a design tool was in the estimation of transient behavior (Kalman and Bertram 1960, Letov 1961). In particular, it was used to obtain an approximation of the settling time. By making this approximation of the settling time as small as possible, it was argued that the speed of response would be increased. Johnson (1963) has recently employed such an approach for the design of a class of sub-optimal control systems.

Consider a positive definite scalar function, $V(x)$, whose total time derivative, $\dot{V}(x)$, is negative definite. Then by the use of the Second Method, one may conclude asymptotic stability of the origin. However, although one knows that the motion tends toward the origin, the rate at which the origin is approached is unknown. Now define η as

$$\eta = \min_x \frac{-\dot{V}(x)}{V(x)} \quad (A.1)$$

Then

$$\dot{V}(x) \leq -\eta V(x) \quad (A.2)$$

which may be solved to give

$$V(x(t)) \leq V(x(0))e^{-\eta t} \quad (A.3)$$

Thus, given the value of $V(x)$ at $t = 0$, an upper bound on the value of $V(x(t))$ at any time $t > 0$ can be obtained by the

use of Equation (A.3). Therefore from the initial state x^0 , the state of the system must be found within or on the surface $V(x) = V(x^0)e^{-\eta t_1}$ after t_1 seconds. For an illustration of how this procedure can be used to estimate settling time, consider the following example.

Example A.1 The equations of motion for the system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (\text{A.4})$$

It is desired to find an upper bound on the time that it takes the system to get from the initial condition $x^0 = (1,0)$ to within the area defined by $(x_1)^2 + (x_2)^2 \leq 0.01$.

In this case it is necessary to find the largest value of K such that the surface $V(x) = K$ lies entirely within or at most tangent to the surface $(x_1)^2 + (x_2)^2 = 0.01$. See Figure A.1. Then by the use of Equation (A.3), the settling time, t_s , is

$$t_s = \frac{-1}{\eta} \ln \left(\frac{K}{V(x^0)} \right) \quad (\text{A.5})$$

However, before this can be done, it is necessary to find η .

Let $\dot{V}(x)$ be defined by the quadratic form $\dot{V}(x) = -x'Qx$ where Q is a symmetric positive definite matrix. Then $V(x)$ is the quadratic form $V(x) = x'Px$ where P is a positive definite symmetric matrix which is the unique solution of the matrix equation.

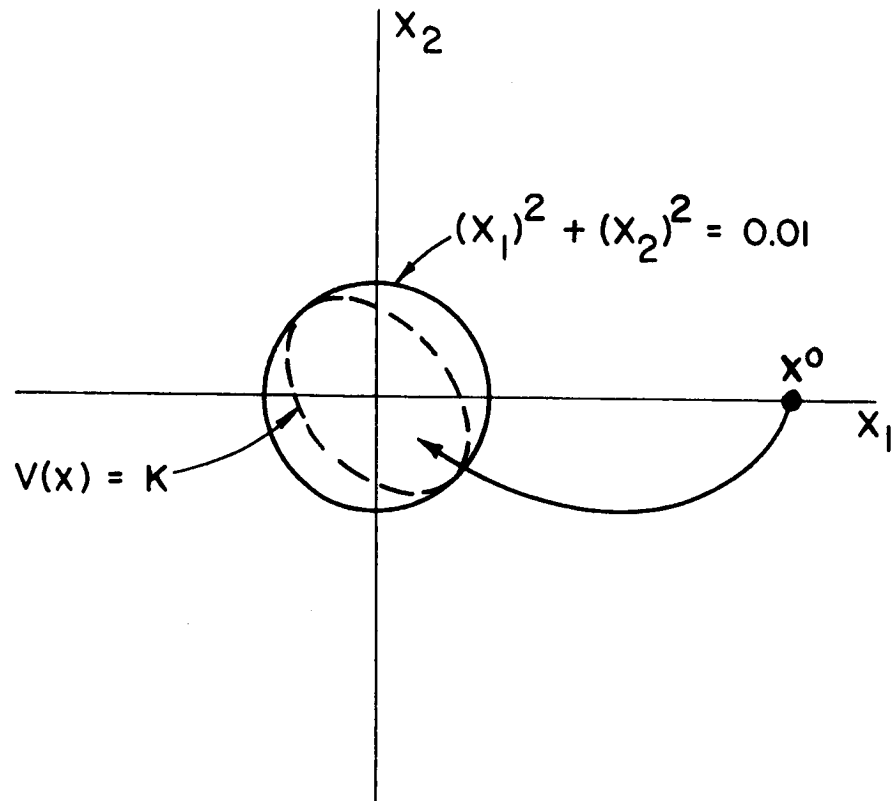


Figure A.1 Estimation of Settling Time

$$\underline{A}'\underline{P} + \underline{P}\underline{A} = -\underline{Q} \quad (\text{A.6})$$

Kalman and Bertram (1960) have shown that η is given by

$$\eta = \text{minimum eigenvalue of } \underline{Q}\underline{P}^{-1}$$

Now let \underline{Q} be

$$\underline{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

Then by the use of Equation (A.6), \underline{P} is given by

$$\underline{P} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

and η is equal 0.775.

For this $V(x)$, K is found to be 7.64×10^{-3} . The settling time as given by Equation (A.5) is

$$\begin{aligned} t_s &= \frac{-1}{0.775} \ln \left(\frac{7.64 \times 10^{-3}}{5} \right) \\ &= 8.35 \text{ seconds} \end{aligned}$$

This method of estimating the transient behavior of systems has several disadvantages. First, the method is approximate and no knowledge of the quality of the approximation is known. Second, the value of η and hence t_s depends on the particular $V(x)$ used. No method of picking $V(x)$ is known. Third, it is necessary that $\dot{V}(x)$ be

negative definite. This is very difficult to attain in practice except for linear systems.

A.3 Kalman-Bertram Method

In 1960 Kalman and Bertram presented a method for designing approximately time-optimal control systems. Their method was based on the knowledge that for a closed, bounded control region, U , the control vector is always on the boundary. They suggested minimizing the time derivative of $V(x)$, arguing that this would make $V(x)$ approach zero most rapidly, and the state of the system should reach the origin in minimum time.

Consider the system

$$\dot{x} = \underline{A}x + \underline{B}u \quad (\text{A.7})$$

where the control region U is defined by the set of all control vectors u such that $|u_i| \leq M_i$, $i = 1, 2, \dots, n$ and M_i are positive constants. Choose arbitrarily a positive semi-definite matrix, \underline{Q} , and then find the positive definite matrix, \underline{P} , which is the unique solution of the matrix equation

$$\underline{A}'\underline{P} + \underline{P}\underline{A} = -\underline{Q} \quad (\text{A.8})$$

Now let $V(x)$ be defined by $V(x) = x'\underline{P}x$ and $\dot{V}(x,u)$ is

$$\dot{V}(x,u) = -x'\underline{Q}x + 2u'\underline{B}'\underline{P}x \quad (\text{A.9})$$

In order to minimize $\dot{V}(x,u)$ with respect to all admissible controls, it is necessary to minimize the second term in $\dot{V}(x,u)$. To minimize this term, each component of u must have its maximum magnitude in the direction opposite that of the corresponding component of $\underline{B}'\underline{P}x$. Therefore

$$u_1 = -M_1 \text{sgn} \left[(\underline{B}'\underline{P}x)_1 \right] \quad (\text{A.10})$$

As an illustration of this procedure consider the following example.

Example A.2 The equations of motion of the system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

It is desired to drive this system to the origin from any initial state in minimum time.

The first step in the procedure is an arbitrary choice of \underline{Q} . In this case let \underline{Q} be

$$\underline{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

in which case \underline{P} as obtained from Equation (A.8) becomes

$$\underline{P} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then by the use of Equation (A.10) the control vector components are found to be

$$u_1 = -M_1 \operatorname{sgn} (2x_1)$$

$$u_2 = -M_2 \operatorname{sgn} (2x_1 + x_2)$$

This method has several advantages. First, it provides a closed-loop solution of very simple form. The method is relatively easy to apply to high-order and multiple input systems. The control matrix \underline{B} is not required to be non-singular.

The main disadvantage of the method is the lack of a procedure for choosing the \underline{Q} matrix and hence $V(x)$. Since the solution depends strongly on $V(x)$, it would be highly desirable to have a procedure for choosing the "best" $V(x)$ or at least an iterative method for improving on an initial choice. Again the method provides only approximately optimum performance and no procedure for evaluating the quality of the approximation is presented. The resulting sub-optimal control system normally experiences chattering near the origin, which degrades its performance.

One additional point should be mentioned with regard to the choice of quadratic forms. If $V(x)$ is chosen to be a quadratic form, then, in general, the "best" $V(x)$ depends on the initial state of the system. Hence one needs some form of weighting of the state space with regard to possible initial states before the "best" $V(x)$ can be selected. Such a weighting would be extremely difficult to realize in practice.

A.4 The Nahi Method

Nahi (1964) has recently presented a method of designing sub-optimal control systems based on the concept of forcing

$$\min_{u \in U} \dot{V}(x, u) \leq -K_1 V(x) - 2K_2 \sqrt{V(x)} \quad (A.11)$$

This method was based on two arguments. First, minimizing $\dot{V}(x, u)$ would minimize the response time. Second, forcing minimum $\dot{V}(x, u)$ to be less than or equal to $-K_1 V(x) - 2K_2 \sqrt{V(x)}$ would make the response time finite, as is shown below.

The systems to be considered must be represented in the following form

$$\dot{x} = \underline{A}x + \underline{B}u \quad (A.12)$$

where \underline{B} is a non-singular matrix and the control region U is defined by the set of all control vectors u such that $\|u\| \leq 1$. Choose arbitrarily a positive definite matrix \underline{Q} , and find the positive definite matrix, \underline{P} , which is the unique solution of the matrix equation

$$\underline{A}'\underline{P} + \underline{P}\underline{A} = -\underline{Q} \quad (A.13)$$

Now let $V(x)$ be defined by $V(x) = x'\underline{P}x$ and then $\dot{V}(x, u)$ is

$$\dot{V}(x, u) = -x'\underline{Q}x + 2u'\underline{B}'\underline{P}x \quad (A.14)$$

In order to minimize $\dot{V}(x,u)$ with respect to all admissible controls, u must be given by

$$u = \frac{-\underline{B}'\underline{P}x}{\|\underline{B}'\underline{P}x\|} \quad (\text{A.15})$$

Then substituting Equation (A.15) for u into Equation (A.14) gives

$$\min_{u \in U} \dot{V}(x,u) = -x'\underline{Q}x - 2\sqrt{x'\underline{P}\underline{B}'\underline{B}Px} \quad (\text{A.16})$$

Nahi (1964) has shown that there exist two positive constants, K_1 and K_2 , defined by

$$K_1 = \text{minimum eigenvalue of } \underline{Q}\underline{P}^{-1} \quad (\text{A.17})$$

$$(K_2)^2 = \text{minimum eigenvalue of } \underline{P}\underline{B}\underline{B}' \quad (\text{A.18})$$

such that the following conditions are satisfied.

$$1. \quad x'\underline{Q}x \geq K_1 x'\underline{P}x \quad (\text{A.19})$$

$$2. \quad x'\underline{P}\underline{B}\underline{B}'Px \geq (K_2)^2 x'\underline{P}x \quad (\text{A.20})$$

Then substituting Equations (A.19) and (A.20) in Equation (A.16) gives

$$\begin{aligned} \min_{u \in U} \dot{V}(x,u) &\leq -K_1 x'\underline{P}x - 2K_2 \sqrt{x'\underline{P}x} \\ &\leq -K_1 V(x) - 2K_2 \sqrt{V(x)} \end{aligned} \quad (\text{A.21})$$

Now for some given initial state $x(0)$, Equation (A.21) can be solved to obtain

$$\frac{2}{K_1} \ln \left(K_1 \frac{\sqrt{V(x(t))}}{2K_2} + 1 \right) - \frac{2}{K_1} \ln \left(K_1 \frac{\sqrt{V(x(0))}}{2K_2} + 1 \right) \leq -t$$

If $V(x(t))$ is set equal to zero, then t becomes the transition time from $x(0)$ to the origin, t_0 .

$$t_0 \leq \frac{2}{K_1} \ln \left(\frac{K_1 \sqrt{V(x(0))}}{2K_2} + 1 \right) \quad (\text{A.22})$$

Hence the transition time is not only known to be finite, but also an upper bound on it is obtained. As an illustration of the above procedure, consider the following example.

Example A.3 The equations of motion of the system are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (\text{A.23})$$

It is desired to design a sub-optimal control system which transfers the system from any initial state to the origin in a finite time. An upper bound on the transition time should also be obtained.

The first step in the procedure presented above is to arbitrarily choose a \underline{Q} matrix. For this problem let \underline{Q} be

$$\underline{Q} = \begin{bmatrix} 4 & 5 \\ 5 & 10 \end{bmatrix}$$

in which case \underline{P} as obtained from Equation (A.13) is

$$\underline{P} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Then the desired sub-optimal control as given by Equation (A.15) is

$$u = \frac{1}{\sqrt{4(x_1)^2 + 4x_1x_2 + 13(x_2)^2/4}} (2x_1 + x_2, 3x_2/2)$$

From Equations (A.17) and (A.18) the constants K_1 and K_2 are found to be

$$K_1 = 1.0$$

$$K_2 = 1.224$$

Then by the use of Equation (A.22), the upper bound on the transition time is

$$t_0 \leq 2 \ln \left(\frac{\sqrt{2(x_1)^2 + 2x_1x_2 + 2(x_2)^2}}{2.45} + 1 \right)$$

This method has two serious disadvantages. First, the control matrix, \underline{B} , must be non-singular. This, in general, is not true in practice. If \underline{B} is singular, then K_2 is zero, and the transition time is infinite. Second, as pointed out in the previous section, there is no procedure for choosing the "best" \underline{Q} matrix.

On the other hand, the method does provide a relatively simple closed-loop solution. The transition time is finite and an upper bound on it is readily obtained. However, there is no means of judging how close the transition time of the sub-optimal system is to the optimum.

LIST OF REFERENCES

- Boltyanskii, V. G., R. V. Gamkrelidze, and L. S. Pontryagin, "The Theory of Optimal Processes. I. The Maximum Principle," Izv. Akad. Nauk. SSSR, Ser. Mat., Vol. 24 (1960), 3-42.
- Hahn, Wolfgang. Theory and Application of Liapunov's Direct Method. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1963.
- Horowitz, Isaac M. Synthesis of Feedback Systems. New York: Academic Press, 1963.
- Johnson, George W. Design of Optimal Control Systems with Stability Constraints Via the Direct Method of Liapunov. Storrs, Connecticut: University of Connecticut, Ph.D. Thesis, 1963.
- Kalman, R. E. and J. E. Bertram, "Control System Analysis and Design Via the 'Second Method' of Liapunov, Part I, Continuous-Time Systems," Trans. of ASME, J. of Basic Engr. (June 1960), 371-393.
- Kalman, R. E., "The Theory of Optimal Control and the Calculus of Variations," RIAS Technical Report No. 61-3, 1961.
- Krassovskii, N. N., "On the Theory of Optimum Control," PMM, Vol. 23, No. 4 (1959), 899-918.
- LaSalle, J. P., "Some Extensions of Liapunov's Second Method," IRE Trans. on Circuit Theory, Vol. 7 (December 1960), 520-27.
- LaSalle, J. P. and Solomon Lefschetz. Stability by Liapunov's Direct Method with Applications. New York: Academic Press, 1961.
- LaSalle, J. P., "Stability and Control," J. SIAM Control, Series A, Vol. 1, No. 1 (1962), 3-15.
- Letov, A. M. Stability in Nonlinear Control Systems. Princeton, N. J.: Princeton University Press, 1961.
- Malkin, I. G. Theory of Stability of Motion. Washington, D. C.: AEC Translation No. 3352, 1958.

- Mohler, R. R. Optimal Control of Nuclear-Reactor Processes. Ann Arbor, Michigan: University of Michigan Ph.D. Thesis, 1965.
- Mohler, R. R. and J. E. Perry, Jr., "Nuclear Rocket Engine Control," *Nucleonics*, Vol. 19, No. 4 (April 1964).
- Nahi, N. E., "On Design of Time Optimal Systems Via the Second Method of Liapunov," University of Southern California, School of Engr. USCEE Report No. 103, January 1964.
- Pontryagin, L. S., et. al. The Mathematical Theory of Optimal Processes. New York: Interscience, 1962.
- Schultz, D. G. The Variable Gradient Method of Generating Liapunov Functions, with Applications to Automatic Control. Lafayette, Indiana: Purdue University Ph.D. Thesis, 1962.
- Truxal, J. G. Automatic Feedback Control System Synthesis. New York: McGraw-Hill Book Company, Inc., 1955.